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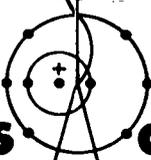
# Pi-Meson Absorption on the Deuteron

by

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## PI-MESON ABSORPTION ON THE DEUTERON

Bruce Goplen, Ph.D.

The nonradiative absorption of positive pi-mesons by deuterons is studied within the framework of Lagrangian formalism and the impulse approximation. The pi-nucleon absorption interaction is taken to be pseudovector; however, the reaction is believed to be dominated by pi-nucleon scattering. Thus, the main thrust of the investigation is directed at three different approaches to the scattering interaction.

First, the Hamiltonian formalism of Koltun and Reitan is extended to include energetic pions. An off-shell form is chosen for p-wave scattering which at low energy reduces to the field-theoretical result of Klein. In the second approach, the S-matrix formalism of Lazard, Ballot and Becker is adapted for use with phenomenological nucleon-nucleon wave functions. Here the scattering interaction off-shell is given by the on-shell amplitudes. Finally, a pion optical model formalism is presented in which scattering graphs are included through modification of the external pion wave function. A new second-order optical model for the deuteron is given.

In all three approaches calculations are performed using phenomenological nuclear potentials, including the Boundary Condition Model and the Hamada-Johnston.

Calculated results for total and differential cross sections are compared with experimental data from 2 to 240 MeV pion kinetic energy.

The S-matrix approach and two forms of the Hamiltonian method are successful in representing the data, while the Kisslinger optical model produces a resonance below the experimental peak. Results are also presented for pi-deuteron elastic scattering calculations using the pion optical model, which gives reasonable agreement with the experimental data.

In conclusion, it does not appear possible to confirm the Galilean-invariant absorption term with this reaction. However, differential cross section results exhibit some sensitivity to nuclear models and to the D-state fraction in the deuteron. An experiment now under way at LAMPF may allow selection of preferred nuclear models. While use of an on-shell scattering Hamiltonian gives the desired total cross section, this result is not credible since calculations show great sensitivity to the off-shell behavior. The pi-deuteron absorption reaction offers a good means of measuring this off-shell behavior. Finally, a separable Hamiltonian interaction density is recommended for the investigation of nuclear structure using pi-meson absorption.

CHAPTER 1  
INTRODUCTION

The absorption of pi-mesons on nuclei has proven a subject of considerable interest, both theoretical<sup>1-49</sup> and experimental.<sup>50-73</sup> Particularly important is the reaction leading to the emission of two nucleons, according to



The reason for interest in this reaction is that kinematical considerations indicate a sensitivity to the short-range nucleon-nucleon interaction. For example, in the nuclear absorption of a low-energy pi-meson, the outgoing nucleon(s) must have total kinetic energy roughly equal to the mass of the pion. The momentum required to make this reaction go cannot come from the low-energy pion, but must instead be found within the nucleus. This momentum is greater than that generally associated with Fermi momenta, so that single-nucleon emission is a relatively unlikely process. On the other hand, the required momentum may be produced by a correlated pair, and is associated with a very short-range interaction. Absorption of the pi-meson on the correlated pair then leads to two-nucleon emission, preferentially back-to-back, from the nucleus. This process may be represented by the equation,





Two-nucleon emission is thus, in principle, a reaction capable of testing the nuclear correlation function. The possible utility of the pi-meson in exploring nuclear structure was first suggested by Brueckner,<sup>1</sup> who noted the dominance of the reaction (1-1) and proposed the correlated pair hypothesis.

Of the general class of reactions (1-1), the process,



would seem to be of special interest, since it is more amenable to analysis than processes involving complicated nuclei. The solution of the deuteron two-body problem using Schrödinger's equation has been much studied, often in conjunction with nucleon-nucleon scattering. Thus both initial and final nuclear states in this reaction are relatively well known. Of particular importance is the absence in the final state of a residual nucleus,  $A^*$ , which is believed to distort the outgoing nucleon-nucleon wave function.<sup>35</sup> Similarly, in the case of the deuteron there is no nuclear distortion of the incident pion wave function, except by the absorbing pair.

For these reasons it was hoped that a credible calculation could be performed for pion absorption on the deuteron. Without an adequate treatment of this process, one could not express confidence in the application of similar techniques for the extraction of nuclear structure information.

Much experimental data<sup>62-72</sup> is available for the absorption of pi-mesons on deuterons and the inverse reaction, production. These reactions can be related through detailed balance. Generally, we have emphasized the absorption cross section,  $\sigma_a$ , vs  $q$ , the incident pion momentum in the

overall-center-of-mass frame, or vs  $E_\ell$ , the pion kinetic energy in the laboratory frame. The first choice tends to more clearly illustrate the  $q^{-1}$  behavior of the absorption cross section at low energy and to emphasize these data. After a minimum near 20 MeV, the total cross section appears to rise linearly with momentum to a prominent resonance which is centered near 140 MeV. The cross section then falls off rapidly just above the resonance.

The differential cross section is symmetric front-to-back in the overall-center-of-mass frame. In the low-energy region the angular distributions appear nearly isotropic, while near the resonance, distributions are strongly anisotropic with peaking in the forward and backward directions. In addition to total and differential cross sections, some polarization data are available. A recent experiment at CERN<sup>72</sup> has obtained differential cross sections at several higher energies. An experiment now under way at LAMPF<sup>73</sup> involves differential cross section measurements at pion energies between 10 and 60 MeV.

Early experiments involving the absorption reaction and its inverse helped to establish the spin and parity of the pi-meson. Consideration of parity conservation, conservation of angular momentum and nucleon-nucleon antisymmetry allow selection of final-state quantum numbers given the incident pion angular momentum. Table I presents allowed transitions for the first five partial waves in the expansion of the pion wave function. These transitions may be considered to occur from either the  $^3S_1$  or the  $^3D_1$  state of the deuteron.

The proton-proton states given in Table I describe coordinate-space distributions with respect to the relative momentum vector,  $\vec{k}$ . The

Table I

Allowed Transitions from the Deuteron  ${}^3S_1$  and  ${}^3D_1$  States

<u>Pion Partial Wave</u>	<u>Proton-Proton State</u>
s	${}^3P_1$
p	${}^1S_0, {}^1D_2$
d	${}^3P_1, {}^3P_2 - {}^3F_2, {}^3F_3$
f	${}^1D_2, {}^1G_4$
g	${}^3F_3, {}^3F_4 - {}^3H_4, {}^3H_5$

experimental measurement, on the other hand, involves a distribution in  $\theta$ , which is defined as the angle between the nuclear vector,  $\vec{k}$ , and the incident pion momentum vector,  $\vec{q}$ . It can be shown that this distribution carries components reflecting the incident pion partial wave. For example, the final state,  ${}^3P_1$ , resulting from the initial state,  $s$ - ${}^3S_1$ , is isotropic in the overall-center-of-mass frame.

We now review some of the earlier theoretical treatments of this reaction. One of the first recorded is the work of Tamor,<sup>40</sup> who considered the decay of a mesic atom, which consists of a negative pi-meson bound to the deuteron by Coulomb forces. This work helped to establish the pi-meson as a pseudoscalar or pseudovector particle, that is, one of intrinsic negative parity. Another early calculation was that of Brueckner,<sup>39</sup> who calculated coefficients for the low-energy expression,

$$\sigma_a = A\bar{q}' + Bq. \quad (1-4)$$

This form well represents the experimental absorption data below the resonance. Physical arguments suggest that the first term is representative of the pion s-wave, while the second is a p-wave term. As will be shown in Chapter 2, the production and absorption reactions can be related precisely by detailed balance. At low pion energy the production cross section is very nearly proportional to the absorption cross section times the pion momentum squared. Thus for production, the expression analogous to Eq. (1-4) contains terms which are linear and cubic in the pion momentum. Many of the early efforts sought to calculate one or both of these coefficients. The relativistic approach of D. Schiff,<sup>46</sup> for example, obtained reasonable agreement for the  $q^3$  behavior on the low-energy side of the production resonance.

The majority of calculations, however, have been based upon Lagrangian formalism. Since the new calculations to be reported here fall into this category, a brief review is given of preceding efforts of this type. First, we wish to make clear the distinction between pi-nucleon operators and the pi-deuteron angular momentum, or partial wave. This problem arises since use is frequently made of the terms s-wave and p-wave in describing operators in the absorption Hamiltonian. These designations refer to the effect of the operator on the pion field. Thus a p-wave term contains a gradient operating on the pion field, while an s-wave term contains a gradient operating on the nucleon wave function. This latter term is also commonly referred to as the Galilean-invariant term. The words s-wave and p-wave are also used to refer to the form of the pi-nucleon scattering Hamiltonian. These, too, must be distinguished from pi-deuteron partial waves. Generally, the distinction will be clear from the context.

One of the earliest Lagrangian calculations was performed by Geffen,<sup>41</sup> who evaluated the differential cross section for the production reaction. Figure 1 shows a direct absorption graph for the production process.

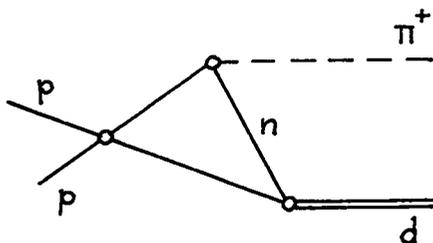


Fig. 1. Direct absorption graph for the production reaction.

Geffen calculated transition matrix elements of the form,

$$m = \langle \psi_f | T | \psi_i \rangle , \quad (1-5)$$

where  $\psi_f$  and  $\psi_i$  are nonrelativistic wave functions describing the final and initial two-nucleon states. The nuclear operator,  $T$ , was taken to be linear in the pseudoscalar pion field; s- and p-wave operator terms were successfully parameterized independently of energy. The initial nuclear states,  $^3P_1$ ,  $^1S_0$  and  $^1D_2$ , were obtained by solution of the Schrödinger equation using Jastrow potentials, while the deuteron wave function consisted of a combination of central and tensor Yukawa wells. Geffen obtained reasonable agreement with experimental data near threshold; his model suggested the importance of the nuclear interaction, in particular, a repulsive core, and inclusion of the D-state of the deuteron.

A similar calculation was performed by Lichtenberg,<sup>42</sup> who also calculated matrix elements using wave functions of physical nucleons. He considered a Hamiltonian linear in both pion field and pion momentum. The  $^1S_0$  and  $^1D_2$  states were then used to calculate the contribution to p-wave pions. The wave functions for the proton-proton states as well as those for the deuteron were obtained by solution of Schrödinger's equation using the Gartenhaus potential. This calculation gave qualitative agreement for the  $q^3$  term in the production cross section near threshold.

In a later publication, Lichtenberg<sup>43</sup> demonstrated the role played by pi-nucleon scattering in the production reaction. In this calculation, a process was included in which the pi-meson, following production at the first nucleon, scatters off the second prior to emission. This matrix element thus included an integral over the intermediate-state momentum of an off-shell transition matrix for the (3,3) scattering state. The off-shell form was chosen to reduce on-shell to the free scattering amplitude. Results showed improved agreement with the  $q^3$  experimental term. Lichtenberg also suggested the need for experiments near threshold to better determine the role of s-wave pions in this reaction.

Woodruff<sup>44</sup> performed a calculation for the production reaction near threshold in which both s- and p-wave scattering processes were included along with the direct production graph shown in Fig. 1. The scattering processes included forward and backward propagation in time, as illustrated by the graphs in Fig. 2. Meson production was considered to occur through the static p-wave term and an s-wave term chosen to make the production operator Galilean-invariant. Gartenhaus deuteron and Gammel-Thaler scattering wave functions were used. Woodruff found it possible to fit the low-energy experimental data with the exception of the pi-deuteron

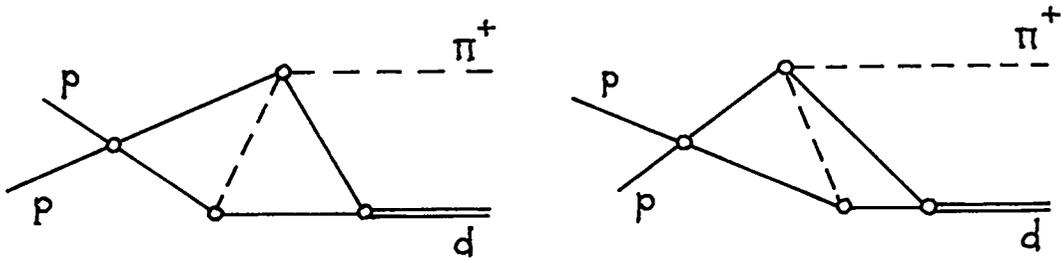


Fig. 2. Scattering graphs for the production reaction.

s-wave term. He pointed out that the Galilean-invariant contribution to this s-wave production was inadequate at threshold, due to cancellation of deuteron S- and D-state terms, but that s-wave pi-nucleon scattering (Fig. 2) could give a contribution of sufficient magnitude.

The general approach of Woodruff was also taken by Koltun and Reitan,<sup>45</sup> who obtained better agreement for the pion s-wave reaction. They considered specifically the absorption of a bound, negative pi-meson by the deuteron. Direct s-wave absorption as well as s-wave scattering on either nucleon followed by s- and p-wave absorption on the other were included. Their treatment differed from that of Woodruff mainly in the relative phase of the forward and backward terms involving propagation of the virtual meson. This factor and use of the Hamada-Johnston potential allowed better agreement with data at threshold. No important difference was found, however, when the Gammel-Thaler scattering wave functions and Gartenhaus deuteron were used. A significant contribution was obtained from the charge-exchange scattering term. Koltun and Reitan confirmed the approximate cancellation of the deuteron S- and D-state direct absorption graphs and thus demonstrated the dominance of scattering graphs even at low pion energies.

This bound-state calculation was later extended to include low-energy pions by Reitan,<sup>48</sup> who included s- and p-waves in the external pion wave function along with s- and p-wave scattering. New coupling constants were used in an attempt to gain agreement with the more recent low-energy data of Rose.<sup>71</sup>

Interest persists in the low-energy region. Cheon and Tohsaki<sup>47</sup> have performed a calculation similar in principle to that of Geffen, using Hamada-Johnston wave functions for the deuteron and an asymptotic form for the proton-proton  $^3P_1$  wave function. They derived a form for the pi-nucleon scattering matrix assuming that the static approximation is applicable. Cheon and Tohsaki considered the basic absorption operator to be p-wave, and added to this an s-wave operator with a separate coupling constant as a free parameter. Their main objective was to determine the value of this coupling constant at low energy. They were not, however, able to achieve agreement with experiment using the usual theoretical reduction of the assumed pseudoscalar interaction. Instead, their parametric studies involving the separate coupling constant indicated close agreement with the earlier result of Geffen, who had considered only direct absorption.

A recent production calculation based on Lagrangian formalism has been done by Lazard, Ballot and Becker.<sup>49</sup> It is notable for the extent of agreement with experiment through the resonance region. The main feature of their S-matrix formalism is the inclusion of on-shell free scattering amplitudes directly in the matrix element. In their approximation I, Lazard et al. considered only s- and p-waves of the outgoing pion, and calculated Born terms and then corrections to these from proton-proton



scattering. In approximation II, the remaining proton-proton partial waves were added in Born approximation. Several deuteron pole models were used; proton-proton scattering was incorporated by modifying the irregular part of an asymptotic wave function with an exponential cutoff. By parameterizing this cutoff function, Lazard et al. attempted to consider the effects of variation in proton-proton short-range behavior. They found the final state interaction to affect the cross section significantly; similarly, sensitivity was found to the choice of deuteron wave function. They were able to compare angular distributions with the recent data from CERN, but were unable to achieve good agreement with the  $\cos^4\theta$  behavior. Finally, they suggested that their calculation be redone with wave functions determined using phenomenological potentials.

We turn now to a brief discussion of the work to be presented here. The main points of interest in pi-deuteron absorption have been demonstrated to involve the nature of the absorption interaction, the nucleon-nucleon interactions, both initial and final, and the pi-nucleon scattering interaction. We generally neglect the question of the absorption interaction and assume the correct operator to be given by the non-relativistic reduction of the pseudovector interaction. Sensitivity to the nucleon-nucleon interactions is explored using realistic wave functions derived by solution of the Schrödinger equation. Phenomenological potentials used include the Hamada-Johnston and several cases of the Boundary Condition Model which represent different fractions of the deuteron D-state. However, the main thrust of this work focuses upon the nature of pi-nucleon scattering as reflected in the absorption reaction. Three separate approaches are taken to the absorption problem, each reflecting a different interpretation of pi-nucleon scattering.

In Chapter 2, the method of Koltun and Reitan is extended to include energetic pions. The p-wave scattering Hamiltonians are taken to be separable, off-shell forms which reduce at low energy to the field-theoretical results given by Klein. A free parameter in these forms reflects the pi-nucleon interaction distance. Still another p-wave Hamiltonian is created to correspond to the on-shell approximation.

In Chapter 3 we follow the suggestion of Lazard et al. and adapt their nuclear second-quantization approach to the more physical wave functions. At low energy these first two approaches will be seen to be nearly equivalent. However, important differences occur in the treatment of the p-wave scattering.

Finally, in Chapter 4 a somewhat different approach is offered in which terms nonlinear in the pion field are deleted from the Hamiltonian. Instead, we choose to modify the field in the remaining direct absorption terms by means of an optical model. Thus, the problem of pi-deuteron scattering is first solved using local, Laplacian and Kisslinger forms of the pion optical model. A new, second-order scattering formalism for the deuteron is offered and found to satisfactorily reproduce the elastic scattering data. With appropriate modification the resulting pion wave functions are used to replace the external pion field in the absorption problem. Direct and charge exchange processes are considered; however, the backwards propagation graphs and the spin-flip scattering terms are omitted.

Within the framework of each of these formalisms, we attempt to calculate the complete matrix element. Thus, all incident pion partial waves are included, and the treatment of the operators, angular momentum relations and integrals is without approximation, other than that

associated with numerical methods. Calculated results for total and differential cross sections are compared with experimental data from low energy through the resonance region. Chapter 5 summarizes the observations and conclusions.

CHAPTER 2  
THE HAMILTONIAN FORMALISM

This work follows closely the theoretical approach of Koltun and Reitan,<sup>45</sup> who calculated the rate of absorption of a negative pion bound by Coulomb forces to the deuteron in an s-orbital. In their calculation, the Hamiltonian interaction density was evaluated between the initial pion state,  $|n,s\rangle$ , and the final vacuum state,  $|0\rangle$ . The resulting transition operator was then taken between initial and final nuclear states. Since only the pion s-orbital was considered, the  ${}^3S_1$  and  ${}^3D_1$  initial states contributed only to the  ${}^3P_1$  final state.

We desire to extend this theory to include the absorption of energetic, positive pi-mesons by deuterons. It is therefore necessary to include higher pion partial waves; these will contribute to the final nuclear states previously given in Table I. The direct absorption p-wave operator must be retained, and, even for low incident pion energies, p-wave scattering terms must be added to the Hamiltonian. Several formulations of this scattering interaction are attempted.

Following the method of Koltun and Reitan, we calculate the nuclear transition operator,

$$T = \sum_{l=0}^{\infty} T^l,$$

where

$$T^0 = \langle 0 | \sum_{i=1}^2 H_i^0 | \bar{q}, c=+1 \rangle$$

and

$$T^\ell = \langle 0 | \sum_{i \neq j} H_i^0 (E - H + i\epsilon)^{-1} H_j^\ell | \bar{q}, c = +1 \rangle, \quad 1 \leq \ell \leq 4. \quad (2-1)$$

Thus the Hamiltonian interaction density is evaluated between the final vacuum state and an initial state which contains a positive pion of momentum,  $q$ .  $T^0$  describes direct absorption while the four other transition operators include pi-nucleon scattering processes. In the scattering propagator,  $E$  is the total initial energy, while  $H$  is the total energy operator for the intermediate state.

These nuclear transition operators,  $T^\ell$ , may be represented by graphs of the kind shown in Fig. 3. The first graph represents the direct absorption process, while the second and third reflect an intermediate pi-nucleon scattering. The third graph is included to permit the virtual meson to scatter backwards in time. The labels indicate particle identities for direct- and charge-exchange scattering processes.

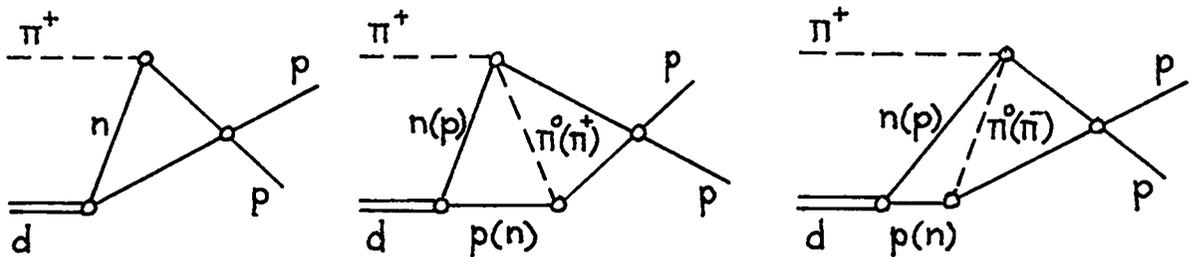


Fig. 3. Direct and scattering graphs for the absorption reaction.

The Hamiltonian interaction density for absorption is taken to be the nonrelativistic reduction of the pseudovector interaction, which is given by

$$H_j^0 = (4\pi)^{1/2} f \bar{\mu}^{-1} i \bar{\sigma}_j \cdot \left\{ i \bar{q}_\pi \tau_j \cdot \phi_j + (2M)^{-1} (\bar{P}_j \tau_j \cdot \pi_j + \tau_j \cdot \pi_j \bar{P}_j) \right\}. \quad (2-2)$$

In this equation,  $\phi_j$  and  $\pi_j$  are the pion and conjugate fields, respectively, evaluated at the coordinate of the  $j^{\text{th}}$  nucleon. These fields couple with and are treated as vectors in isospin.

In this and subsequent chapters, subscripts are used to refer to the nucleon number, while superscripts describe vector components. Generally, use is made of tensor definitions in describing components of vector quantities; the exceptions to this are isospin and the pion field.  $\sigma_j$ ,  $\tau_j$ ,  $\chi_j$  and  $P_j$  refer, respectively, to the spin, isospin, spatial coordinate and momentum of the  $j^{\text{th}}$  nucleon. The symbol,  $q_\pi$ , represents the momentum operator on the pion field, while  $q$ , without sub- or superscript, refers to the external pion momentum. The prime superscript is used to designate intermediate-state meson and final-state nucleon variables. The zero subscript on a momentum variable denotes the zeroth component of four-momentum, or energy. All momenta are measured in the overall-center-of-mass frame. Finally, the symbols  $\mu$  and  $M$  are used for the rest masses of the pion and nucleon, respectively. We do not distinguish the mass of the proton from that of the neutron.

The first term in Eq. (2-2) contains a p-wave operator on the pion field while the remaining terms are s-wave in character with respect to

the field. The sum of the s-wave terms is commonly referred to as the Galilean-invariant term. The operator ordering implies evaluation of nucleon momentum after and before absorption.

The coupling constant used in this and all subsequent chapters is the recently obtained value,<sup>74</sup>  $f^2 = 0.077$ . We point out that  $f$  occurs linearly in all matrix elements, and that the coupling constant may therefore be viewed as an overall normalization factor in the cross section.

The four remaining Hamiltonian terms describe pi-nucleon scattering. For the s-wave interaction, we have taken forms similar to those used by Koltun and Reitan. These are given by Klein<sup>75</sup> as

$$H_f^1 = 4\pi \lambda_1 \bar{\mu}^{-1} \phi_f \cdot \phi_f$$

and

$$H_f^2 = 4\pi \lambda_2 \bar{\mu}^{-2} \tau_f \cdot \phi_f \times \pi_f . \quad (2-3)$$

The quantities,  $\lambda_i$ , are to be determined from the free scattering amplitudes and thus are energy-dependent.  $H^1$  is a direct-exchange operator, while  $H^2$  contains an isospin operator and thus includes charge-exchange terms.

It was suggested in Chapter 1 that the absorption peak is associated with the (3,3) scattering resonance. Therefore it would seem necessary to consider p-wave scattering. Klein has recommended the following p-wave Hamiltonian terms for use at low energy:

$$H_j^3 = -4\pi\lambda_3\bar{\mu}^3(\bar{q}_\pi\phi_j)\cdot(\bar{q}_\pi\phi_j)$$

and

$$H_j^4 = -4\pi\lambda_4\bar{\mu}^3\bar{\sigma}_j\tau_j\cdot(\bar{q}_\pi\phi_j)\times(\bar{q}_\pi\phi_j). \quad (2-4)$$

The dot-cross product in  $H^4$  is understood to apply to the isospin-field operators as well as to the spin-momentum operators.

In calculating matrix elements for the scattering graphs, an integration is performed over all values of the intermediate pion momentum,  $\vec{q}$ . Thus it would seem desirable that the scattering Hamiltonian vanish in the off-shell high-momentum limit. A suitable form for the off-shell behavior has recently been suggested by several authors, including Landau and Tabakin,<sup>76</sup> Myhrer and Koltun,<sup>77</sup> and Eisenberg, Hufner and Moniz.<sup>78</sup> We use this functional form to modify the Klein p-wave terms, according to

$$H_j^3 = -4\pi\lambda_3\bar{\mu}^3(\alpha^2+q^2)^2\left(\frac{\bar{q}_\pi}{\alpha^2+q^2}\phi_j\right)\cdot\left(\frac{\bar{q}_\pi}{\alpha^2+q^2}\phi_j\right)$$

and

$$H_j^4 = -4\pi\lambda_4\bar{\mu}^3(\alpha^2+q^2)^2\bar{\sigma}_j\tau_j\cdot\left(\frac{\bar{q}_\pi}{\alpha^2+q^2}\phi_j\right)\times\left(\frac{\bar{q}_\pi}{\alpha^2+q^2}\phi_j\right). \quad (2-5)$$

In these equations,  $\alpha$  is a parameter which can be related to the pi-nucleon interaction distance. It will be treated here as a free



parameter. The expressions in Eq. (2-5) are seen to be equal in the limit as  $\alpha$  goes to infinity to those of Eq. (2-4). They are also equal in the low-energy and low-momentum limit ( $q \ll \alpha$  and  $q_\pi \ll \alpha$ ) where Klein's results are assumed valid. The forms given in Eq. (2-5) correspond to a separable representation of the pi-nucleon T-matrix; therefore we shall refer to Eq. (2-5) as the separable Hamiltonian.

The nuclear transition operator described in Eq. (2-1) is calculated using the direct absorption, s-wave scattering and separable p-wave Hamiltonians given in Eqs. (2-2), (2-3) and (2-5). Details of this calculation are given in Appendix I. Before stating the result, we mention the desirability of working in center-of-mass and relative nuclear coordinates. The transition operator is then cast as a two-body operator. For coordinate space and momentum, the new variables may be obtained by the transformations,

$$\bar{X}_{1,2} = \bar{R} \pm \frac{1}{2}\bar{r}$$

and

$$\bar{P}_{1,2} = \frac{1}{2}\bar{K} \pm \bar{k}. \quad (2-6)$$

In the overall-center-of-mass frame, the total final-state center-of-mass momentum,  $\bar{K}$ , must vanish. As demonstrated in Appendix I, use of this fact allows a reduction in the number of final integrals which must be evaluated. Treatment of the intermediate state is also discussed in this appendix. The five terms of the nuclear transition operator are given by

$$T^0 = -(4\pi)^{1/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}}$$

$$\left\{ (e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_1^+ \bar{\sigma}_1 + e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_2^+ \bar{\sigma}_2) \cdot \left(1 + \frac{g_0}{2M}\right) \bar{q} \right.$$

$$\left. - (e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_1^+ \bar{\sigma}_1 - e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_2^+ \bar{\sigma}_2) \cdot \left(\frac{g_0}{M}\right) \bar{k}' \right\},$$

$$T^1 = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}}$$

$$\left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_1^+ \bar{\sigma}_1 - e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_2^+ \bar{\sigma}_2 \right) \cdot$$

$$\frac{\int d\bar{q}' e^{i\bar{q}'\cdot\bar{r}} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\}}{(2\pi)^3 \left( \frac{g_0^2}{4} - g_0'^2 \right)},$$

$$T^2 = -\lambda_2 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}}$$

$$\frac{3g_0}{4\mu} (\gamma_1^+ \gamma_2^3 - \gamma_1^3 \gamma_2^+) \left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 + e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \cdot$$

$$\frac{\int d\bar{q}' e^{i\bar{q}'\cdot\bar{r}} \left\{ \left(2 + \frac{g_0^2 + 2g_0'^2}{3g_0 M}\right) \bar{q}' - \frac{2(g_0^2 + 2g_0'^2)}{3g_0 M} \bar{k}' \right\}}{(2\pi)^3 \left( \frac{g_0^2}{4} - g_0'^2 \right)},$$

$$T^3 = -\lambda_3 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}}$$

$$\left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_1^+ \bar{\sigma}_1 + e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \gamma_2^+ \bar{\sigma}_2 \right) \cdot$$

$$\frac{\int d\bar{q}' (\alpha^2 + q^2) e^{i\bar{q}'\cdot\bar{r}} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\} \bar{q}' \cdot \bar{q}}{(2\pi)^3 (\alpha^2 + q^2) \left( \frac{g_0^2}{4} - g_0'^2 \right)}$$

and

$$\begin{aligned}
T^4 = & -\lambda_4 (4\pi)^{3/2} \int \bar{\mu}^{-4} \frac{\sqrt{1}}{\sqrt{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}} \\
& (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \int \frac{d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}'\cdot\bar{r}}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{q_0^2}{4} - g_0^2\right)} \\
& \left\{ e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 \cdot \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\} i\bar{\sigma}_2 \cdot \bar{q}' \times \bar{q} \right. \\
& \left. - e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \cdot \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\} i\bar{\sigma}_1 \cdot \bar{q}' \times \bar{q} \right\}. \quad (2-7)
\end{aligned}$$

The nuclear wave functions used in these calculations are found from a solution of Schrödinger's equation using semiphenomenological nucleon-nucleon potentials. Included are the Hamada-Johnston potential and two cases of the Lomon-Feshbach Boundary Condition Model (BCM) which give different fractions of D-state in the deuteron. A description of the potentials as used in these calculations is given in Appendix II. The potentials include tensor coupling terms, and these lead to coupled, second-order differential equations. These equations are solved numerically using a three point, fifth-order differencing scheme which is developed in Appendix III.

The boundary conditions for the deuteron problem and two different techniques used to obtain eigenvalue solutions are presented in Appendix IV. The initial wave function for each orbital angular momentum state is written as

$$|\psi_i\rangle = (2\pi)^{-3/2} e^{i\bar{k}\cdot\bar{R}} \frac{u_L(r)}{r} |JMLSTM_T\rangle, \quad (2-8)$$

where the angular momentum and isospin variables in the ket refer to the relative nucleon-nucleon system. These variables are, in the order given, total angular momentum and its projection, orbital angular momentum, spin, isospin and projection of isospin. For the deuteron, we have  $J = S = 1$  and  $T = M_T = 0$ . Orbital angular momentum states include the S-state ( $L = 0$ ) and the D-state ( $L = 2$ ). The plane-wave factor describes the motion of the deuteron as an entity.

The proton-proton final state is dealt with in terms of its time-reversed equivalent, the scattering state. The boundary conditions for the tensor-coupled case are obtained using the eigenstate parameterization of Blatt and Biedenharn. This formalism along with symmetrization and time reversal requirements are developed in Appendix V. It will be noted that the final-state wave function is described in terms of helicity states. Thus, the outgoing relative momentum vector,  $\vec{k}$ , becomes an important axis for integration and summing angular momentum components. As shown in Appendix V, a particular component of the final-state wave function may be written as

$$|\psi_f^-\rangle = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{R}} g_{JL'S'}^{M'}(r) |J'M'L'S'T'M_T\rangle, \quad (2-9)$$

where for the singlet and uncoupled-triplet states,

$$g_{JL'S'}^{M'}(r) = \frac{2\sqrt{2\pi}}{k'} (2L'+1)^{1/2} i^{L'} e^{-i\delta_{k'}} \frac{u_{k'}(r)}{r} \begin{bmatrix} J'M' \\ L'O \\ S'M' \end{bmatrix}, \quad (2-10)$$

and for the coupled states,

$$g_{J'L'S'}^{M'}(r) = (a \cos \epsilon_{J'} + b \sin \epsilon_{J'}) e^{-i\alpha} \frac{u_{\alpha, J'+1}(r)}{r} - (a \sin \epsilon_{J'} - b \cos \epsilon_{J'}) e^{-i\beta} \frac{u_{\beta, J'+1}(r)}{r},$$

where

$$a = \frac{2\sqrt{2\pi}}{k'} (2J'-1)^{1/2} i^{J'-1} \begin{bmatrix} J' & M' \\ J'-1 & 0 \\ S' & M' \end{bmatrix}$$

and

$$b = \frac{2\sqrt{2\pi}}{k'} (2J'+3)^{1/2} i^{J'+1} \begin{bmatrix} J' & M' \\ J'+1 & 0 \\ S' & M' \end{bmatrix}. \quad (2-11)$$

The symbol,

$$\begin{bmatrix} J & M \\ L & M_L \\ S & M_S \end{bmatrix},$$

is a convenient notation for the Clebsch-Gordan coefficient; it may be identified with the quantity  $(L S J M | L M_L S M_S)$  defined by Edmonds.<sup>79</sup> The eigenstate symbols,  $\alpha$ ,  $\beta$  and  $\epsilon_{J'}$ , are discussed in Appendix V. For these calculations, the magnitude of the relative momentum vector is determined from energy conservation in the overall-center-of-mass frame. The protons are treated nonrelativistically.

The matrix element for the absorption reaction is obtained by evaluating the transition operator between final and initial nuclear states, according to

$$m = \langle \psi_f | T | \psi_i \rangle. \quad (2-12)$$

In this expression, the implied sums over the nuclear states include total and orbital angular momentum, but not spins or spin projections. Thus, for a given incident pion energy, the matrix element is a function of initial spin projection, final spin and its projection and  $\theta$ , the angle between incident pion and either outgoing proton. The matrix element may be written explicitly as

$$m = \sum_{\lambda=|M'-M|} c(s'M'M\lambda) Y_{\lambda}^{*M'-M}(\hat{k}, \hat{z}). \quad (2-13)$$

The form of Eq. (2-13) is motivated by results developed in Appendix VI and is chosen to provide the differential cross section with a functional dependence on the angle,  $\theta$ . The symbol,  $\lambda$ , is used to denote the external pion partial wave.

Given a matrix element expressed in the form of Eq. (2-13), the differential cross section is obtained by averaging over the initial spin projection and summing over the final spin and its projection, according to

$$\frac{d\sigma_a}{d\Omega} = 2\pi \frac{g_0}{g} \frac{1}{3} \sum_{s'M'M} |m|^2 \frac{v}{(2\pi)^3} \frac{M}{2} k'. \quad (2-14)$$

Since the momenta and the matrix element are determined in the overall-center-of-mass frame, the calculated differential cross section must be associated with this frame also. For historical reasons, it is usually expressed as a polynomial in  $\cos\theta$ -squared, thus directly reflecting the external pion partial waves entering the reaction. As shown in Appendix VII, Eq. (2-14) may be written as

$$\frac{d\sigma_a}{d\Omega} = \frac{MVk'g_2}{96\pi^3g} \sum_{S'M'M} (-1)^{M'-M} \sum_{\lambda, \lambda'=|M'-M|} \sqrt{(2\lambda'+1)(2\lambda+1)} \\ C^*(S'M'M\lambda') C(S'M'M\lambda) \sum_{\ell=|\lambda'-\lambda|}^{\lambda'+\lambda} \begin{bmatrix} \ell & 0 \\ \lambda' & 0 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \ell & 0 \\ \lambda & M'-M \\ \lambda & M-M' \end{bmatrix} a_\ell^{-1} \sum_{n=0}^{\ell} d_\ell^n \cos^n(\theta). \quad (2-15)$$

In this equation,  $a_\ell$  and  $d_\ell^n$  are coefficients which explicitly define the Legendre polynomials in terms of the cosine function. Equation (2-15) can be simplified to obtain the form,

$$\frac{d\sigma_a}{d\Omega} = \sum_{n=0} C_{2n} \cos^{2n}(\theta). \quad (2-16)$$

Since the differential cross section expresses the probability for the arrival of either proton at the target, the total cross section must be given by

$$\sigma_a = \frac{1}{2} \int \frac{d\sigma_a}{d\Omega} d\Omega. \quad (2-17)$$

In practice, the cross section may be obtained directly from Eqs. (2-13) and (2-14) using the orthonormality of the spherical harmonics. The result is

$$\sigma_a = \frac{MVk'g_0}{48\pi^2g} \sum_{\substack{s'M'M \\ \lambda=|M'-M|}} |c(s'M'M\lambda)|^2. \quad (2-18)$$

Alternately, the cosine-squared polynomial of Eq. (2-16) may be integrated to obtain

$$\sigma_a = 2\pi \sum_{n=0} \frac{C_{2n}}{2n+1}. \quad (2-19)$$

The total cross section for the inverse reaction, or production, can be shown using detailed balance to be

$$\sigma_p = \frac{3}{2} \left(\frac{g}{k'}\right)^2 \sigma_a. \quad (2-20)$$

We now return to the evaluation of the matrix element of Eq. (2-13) using results of Appendix VI. The integral over the center-of-mass coordinate,  $\bar{R}$ , yields only a delta function in momentum, according to

$$(2\pi)^{-3} \int d\bar{R} e^{-i\bar{K}'\cdot\bar{R}} e^{i\bar{K}\cdot\bar{R}} e^{i\bar{Q}\cdot\bar{R}} = \delta(\bar{K} + \bar{Q} - \bar{K}'). \quad (2-21)$$



The matrix element is therefore determined solely by integration over the relative coordinate,  $\bar{r}$ . The complex coefficients are broken down into analytic angular integrals and one-dimensional radial integrals which must be performed numerically. These are labeled  $R_i$  and  $I_i$ , respectively. The result is that the coefficient may be written as

$$C(S'M'M\lambda) = i^{\lambda+1} (4\pi)^{3/2} f \bar{\mu}^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} \sum_{J'L'L} \sum_{i=1}^{24} R_i I_i . \quad (2-22)$$

The integrals,  $R_i$  and  $I_i$ , are functions of the initial and final nuclear quantum numbers as well as the incident pion partial wave. Thus, for each partial wave, these integrals must be evaluated for all possible interacting sets of initial and final two-nucleon states. These states have previously been listed in Table I.

The angular integrals,  $R_i$ , are given explicitly in Appendix VI and will not be repeated here since they are used in all three of the formalisms to be presented. These integrals were found to satisfy a Wigner-Eckart result on the Clebsch-Gordan coefficient involving conservation of total angular momentum,

$$\begin{bmatrix} J' & M' \\ J & M \\ \lambda & \mu \end{bmatrix} .$$

Thus, in practice, a momentum projection set involving a nonzero coupling coefficient was used to calculate integrals,  $R_i$ , and a reduced

matrix element. The Wigner-Eckart theorem<sup>80</sup> was then applied to obtain results for all other projections.

The derivation of the integrals,  $R_i$ , leads naturally to the following definitions for the one-dimensional radial integrals:

$$I_1 = -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} - \frac{\lambda}{r}\right) j_\lambda,$$

$$I_2 = -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} + \frac{\lambda+1}{r}\right) j_\lambda,$$

$$I_3 = -\frac{g_0}{M} \int dr r^2 \frac{u}{r} j_\lambda \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*,$$

$$I_4 = -\frac{g_0}{M} \int dr r^2 \frac{u}{r} j_\lambda \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*,$$

$$I_5 = \mu \left(\lambda_1 + \frac{3g_0}{2\mu} \lambda_2\right) \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} f_{31} j_\lambda g'^*,$$

$$I_6 = \mu \left(\lambda_1 + \frac{3g_0}{2\mu} \lambda_2\right) \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} f_{31} j_\lambda g'^*,$$

$$I_7 = -\left(\lambda_1 + \frac{3g_0}{2\mu} \lambda_2\right) \frac{g_0}{M} \int dr r^2 \frac{u}{r} f_{20} j_\lambda \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*,$$

$$I_8 = -\left(\lambda_1 + \frac{3g_0}{2\mu} \lambda_2\right) \frac{g_0}{M} \int dr r^2 \frac{u}{r} f_{20} j_\lambda \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*,$$

$$I_9 = -\lambda_3 g \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{40} j_{\lambda-1} g'^*,$$

$$I_{10} = -\lambda_3 g \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{40} j_{\lambda+1} g'^*,$$

$$I_{11} = -\lambda_3 g \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{42} j_{\lambda-1} g'^*,$$

$$I_{12} = -\lambda_3 g \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{42} j_{\lambda+1} g'^*,$$

$$\begin{aligned}
I_{13} &= \lambda_3 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda-1}^i \left( \frac{d}{dr} - \frac{L'}{r} \right) g'^{*}, \\
I_{14} &= \lambda_3 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda-1}^i \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g'^{*}, \\
I_{15} &= \lambda_3 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda+1}^i \left( \frac{d}{dr} - \frac{L'}{r} \right) g'^{*}, \\
I_{16} &= \lambda_3 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda+1}^i \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g'^{*}, \\
I_{17} &= -\lambda_4 2g \left( 2 + \frac{g_0}{2M} \right) \int dr r^2 \frac{u}{r} \tilde{f}_{40}^{\sim} j_{\lambda-1}^i g'^{*}, \\
I_{18} &= -\lambda_4 2g \left( 2 + \frac{g_0}{2M} \right) \int dr r^2 \frac{u}{r} \tilde{f}_{40}^{\sim} j_{\lambda+1}^i g'^{*}, \\
I_{19} &= -\lambda_4 2g \left( 2 + \frac{g_0}{2M} \right) \int dr r^2 \frac{u}{r} \tilde{f}_{42}^{\sim} j_{\lambda-1}^i g'^{*}, \\
I_{20} &= -\lambda_4 2g \left( 2 + \frac{g_0}{2M} \right) \int dr r^2 \frac{u}{r} \tilde{f}_{42}^{\sim} j_{\lambda+1}^i g'^{*}, \\
I_{21} &= \lambda_4 2 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda-1}^i \left( \frac{d}{dr} - \frac{L'}{r} \right) g'^{*}, \\
I_{22} &= \lambda_4 2 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda-1}^i \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g'^{*}, \\
I_{23} &= \lambda_4 2 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda+1}^i \left( \frac{d}{dr} - \frac{L'}{r} \right) g'^{*}
\end{aligned}$$

and

$$I_{24} = \lambda_4 2 \frac{g}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^{\sim} j_{\lambda+1}^i \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g'^{*}. \quad (2-23)$$

The nature of each integral is easily ascertained from the preceding factor. The first four integrals result from direct absorption, while the next four represent s-wave scattering processes. The remaining

sixteen integrals represent absorption involving p-wave scattering processes. The initial- and final-state wave functions are written simply as  $u$  and  $g'$ , respectively, all other arguments being omitted. The argument of the spherical Bessel function,  $j_\lambda$ , is  $qr/2$ . Although the result of the operator on  $j_\lambda$  in  $I_1$  and  $I_2$  is analytic, we have retained the operator form in preparation for the modified external pion wave function to be used in Chapter 4. The limits of integration over  $r$  are zero to infinity; in practice, the numerical integration was performed from the nucleon-nucleon core out to about 20 fermis.

The variables,  $f_{nl}$  and  $\tilde{f}_{nl}$ , in the radial integrals represent integrals over intermediate pion momentum for s- and p-wave scattering, respectively. These intermediate integrals are defined by the general dimensionless forms,

$$f_{nl}(r) = -\mu^{1-n} \frac{2}{\pi} \int_0^\infty \frac{dq' q'^n j_n(q'r)}{(q'^2 + \mu^2)}$$

and

$$\tilde{f}_{nl}(r) = -(\alpha^2 + q^2) \mu^{1-n} \frac{2}{\pi} \int_0^\infty \frac{dq' q'^n j_n(q'r)}{(q'^2 + \mu^2)(q'^2 + \alpha^2)},$$

where

$$\mu'^2 = \frac{3\mu^2 - q^2}{4}. \quad (2-24)$$

The common factor in the denominators of  $f_{nl}$  and  $\tilde{f}_{nl}$  is simply the product of the forward and backward scattering propagators. Two s-wave intermediate integrals not in the form of Eq. (2-24) are recast in that form to allow grouping of similar terms. In particular, these integrals are

$$-\mu^{-1} \frac{2}{\pi} \int \frac{dq' q'^2 j_0(q'r) \left(1 + \frac{2q_0^2}{q_0^2}\right)}{(q'^2 + \mu^2)} = \frac{3}{2} f_{20}(r)$$

and

$$-\mu^{-2} \frac{2}{\pi} \int \frac{dq' q'^3 j_1(q'r) \left(2 + \frac{q_0}{3M} + \frac{2q_0^2}{3q_0 M}\right)}{(q'^2 + \mu^2)} = \left(2 + \frac{q_0}{2M}\right) f_{31}(r). \quad (2-25)$$

For pion kinetic energies below roughly 180 MeV in the laboratory frame we have  $q^2 < 3\mu^2$ , or  $\mu'^2 > 0$ . In this case, and for  $n + l$  odd, the intermediate integrals are unambiguously obtained by contour integration. The results are

$$f_{nl}(r) = -i^n \left(\frac{\mu'}{\mu}\right)^{n-1} \frac{1}{2} \left\{ h'_l(i\mu'r) + (-1)^n h_l^2(-i\mu'r) \right\}$$

and

$$\begin{aligned} \tilde{f}_{n\lambda}(r) = & -\left(\frac{\alpha^2 + q^2}{\alpha^2 - \mu'^2}\right) \left\{ i^n \left(\frac{\mu'}{\mu}\right)^{n-1} \frac{1}{2} \left\{ h'_\lambda(i\mu'r) + (-1)^n h''_\lambda(-i\mu'r) \right\} \right. \\ & \left. - i^n \left(\frac{\alpha}{\mu}\right)^{n-1} \frac{1}{2} \left\{ h'_\lambda(i\alpha r) + (-1)^n h''_\lambda(-i\alpha r) \right\} \right\}. \end{aligned} \quad (2-26)$$

The required integrals are given more explicitly by

$$\begin{aligned} f_{20}(r) &= -\left(\frac{\mu'}{\mu}\right) \frac{e^{-\mu'r}}{\mu'r}, \\ f_{31}(r) &= -\left(\frac{\mu'}{\mu}\right)^2 \left(1 + \frac{1}{\mu'r}\right) \frac{e^{-\mu'r}}{\mu'r}, \\ \tilde{f}_{31}(r) &= -\left(\frac{\alpha^2 + q^2}{\alpha^2 - \mu'^2}\right) \left\{ \left(\frac{\mu'}{\mu}\right)^2 \left(1 + \frac{1}{\mu'r}\right) \frac{e^{-\mu'r}}{\mu'r} - \left(\frac{\alpha}{\mu}\right)^2 \left(1 + \frac{1}{\alpha r}\right) \frac{e^{-\alpha r}}{\alpha r} \right\}, \\ \tilde{f}_{40}(r) &= \left(\frac{\alpha^2 + q^2}{\alpha^2 - \mu'^2}\right) \left\{ \left(\frac{\mu'}{\mu}\right)^3 \frac{e^{-\mu'r}}{\mu'r} - \left(\frac{\alpha}{\mu}\right)^3 \frac{e^{-\alpha r}}{\alpha r} \right\} \end{aligned}$$

and

$$\begin{aligned} \tilde{f}_{42}(r) &= -\left(\frac{\alpha^2 + q^2}{\alpha^2 - \mu'^2}\right) \\ & \left\{ \left(\frac{\mu'}{\mu}\right)^3 \left(1 + \frac{3}{\mu'r} + \frac{3}{(\mu'r)^2}\right) \frac{e^{-\mu'r}}{\mu'r} - \left(\frac{\alpha}{\mu}\right)^3 \left(1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2}\right) \frac{e^{-\alpha r}}{\alpha r} \right\}. \end{aligned} \quad (2-27)$$

For pion kinetic energies in excess of roughly 180 MeV, we find that  $q^2 > 3\mu^2$ , or  $\mu'^2 < 0$ . In this case, the contour integrals are obtained by raising the positive real pole slightly above the real axis while the negative real pole is lowered. The general results are then given by

$$f_{n\ell}(r) = -i \left( \frac{\mu'}{\mu} \right)^{n-1} \frac{1}{2} \left\{ h'_\ell(\mu'r) + (-1)^n h_\ell^2(-\mu'r) \right\}$$

and

$$\begin{aligned} \tilde{f}_{n\ell}(r) = & - \left( \frac{\alpha^2 + \beta^2}{\alpha^2 \mu'^2} \right) \left\{ i \left( \frac{\mu'}{\mu} \right)^{n-1} \frac{1}{2} \left\{ h'_\ell(\mu'r) + (-1)^n h_\ell^2(-\mu'r) \right\} \right. \\ & \left. - i^n \left( \frac{\alpha}{\mu} \right)^{n-1} \frac{1}{2} \left\{ h'_\ell(i\alpha r) + (-1)^n h_\ell^2(-i\alpha r) \right\} \right\}. \end{aligned}$$

(2-28)

Explicit expressions analogous to those of Eq. (2-27) are easily obtained. The contribution from the pole,  $\mu'$ , in this case reflects an outgoing spherical wave.

We turn now to some exploratory calculations below the resonance and present results obtained using various terms in the Hamiltonian interaction density. At low pion energies, the dimensionless scattering parameters,  $\lambda_i$ , may be determined from results given by Klein. The subscript, 2, indicates evaluation in the two-body (pi-nucleon) frame. The N-star term in the p-wave scattering is neglected to obtain

$$\begin{aligned} \lambda_1 &= -\frac{1}{6} \frac{\mu}{g_2} (\delta_1 + 2\delta_3), \\ \lambda_2 &= \frac{1}{6} \frac{\mu}{g_2} \frac{\mu}{g_2} (\delta_1 - \delta_3), \\ \lambda_3 &= -\left( \frac{\mu}{g_2} \right)^3 \left( \delta_{33} + \frac{1}{2} \delta_{13} \right) \end{aligned}$$

and

$$\lambda_4 = \frac{1}{2} \left( \frac{\mu}{g_2} \right)^3 (\delta_{33} - \delta_{13}). \quad (2-29)$$

In writing these equations, use has been made of the approximation,  $\delta \sim \sin \delta$ . The s-wave phase shifts are designated  $\delta_{2t}$ , while the p-wave phase shifts are given by  $\delta_{2t,2j}$ . For these first calculations, the scattering parameters are not taken to be energy-dependent, but are instead evaluated near zero energy. With the phase shift parameterization of McKinley,<sup>81</sup> the parameters,  $\lambda_i$ , are 0.0054, 0.0445, -0.2407 and 0.1230. However, the energy-dependent results do not change significantly in the first 100 MeV.

Figure 4 presents results using various terms in the Hamiltonian. These calculations were performed over a pion kinetic energy range from 2 to 100 MeV using the Hamada-Johnston potential for initial and final nucleon-nucleon states. The experimental data in Fig. 4 are obtained from references 62, 64-72 and 102.

In the first calculation, only direct absorption processes involving the transition operator,  $T^0$ , are considered. As shown in Fig. 4, the direct absorption result tends nearly to vanish at low energy. This is a consequence of the accidental cancellation of the s-wave absorption operator terms from S- and D-states of the deuteron. This effect has been previously noted by several authors.<sup>44,45,49</sup> The contribution from the p-wave absorption operator, on the other hand, can easily be shown proportional to the incident pion momentum. This is the behavior exhibited by the first curve in Fig. 4.

Addition of the s-wave scattering terms of Eq. (2-3) produces the  $q^{-1}$  behavior which characterizes the low-energy experimental data. This result is shown clearly in Fig. 4. At 2 MeV, a comparison may be made directly with the bound-state calculation of Koltun and Reitan. Six of the integrals in Eq. (2-23), when performed with the  ${}^3S_1$  and  ${}^3D_1$  initial



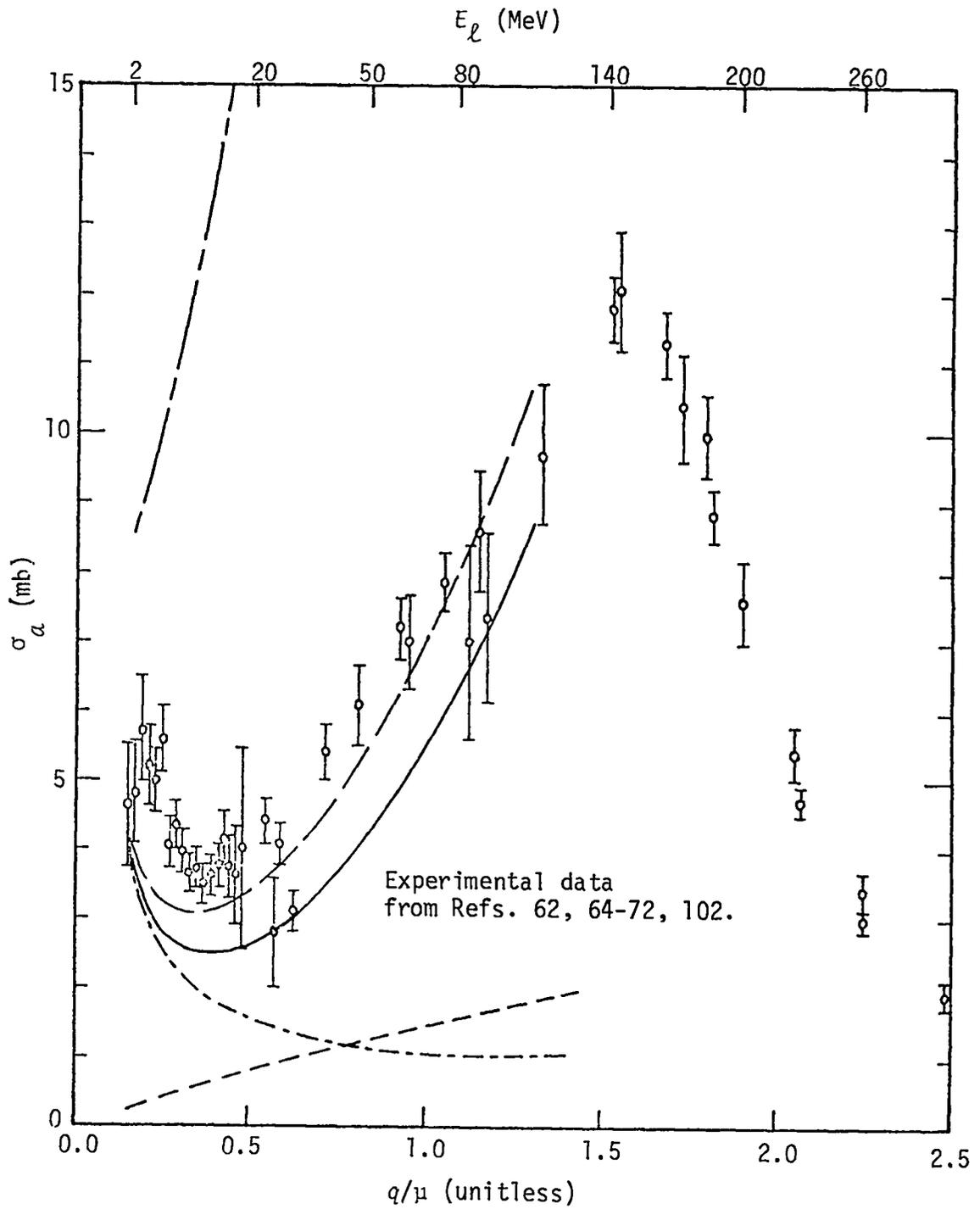


Fig. 4. Cross sections for various Hamiltonian terms.

- direct absorption only
- . - . - . direct absorption and s-wave scattering
- - - - - full calculation with Klein p-wave
- full calculation with on-shell p-wave
- above, with Galilean invariance omitted

and  ${}^3P_1$  final states, differ in form from their integrals only by the factor,  $j_0(\frac{qr}{2})$ . This factor is nearly unity for small pion momentum, so that numerical agreement to within two percent is obtained for all six comparable integrals. No p-wave scattering is considered in this calculation, and as the result in Fig. 4 shows, there is clearly no hope of reproducing the experimental absorption resonance with s-wave scattering processes.

Next, the Klein p-wave scattering terms given in Eq. (2-4) are added to the calculation. Equivalently, the separable forms in Eq. (2-5) may be used with a value of infinity for  $\alpha$ . In this case we have for the intermediate integrals,  $\tilde{f}_{nl} \rightarrow f_{nl}$ . Inclusion of the Klein p-wave terms leads to the excessively large result shown in Fig. 4. This result would seem to reflect the divergent nature of the intermediate integrals, in particular,  $f_{42}$ .

By evaluating these integrals numerically, it is possible to introduce a cutoff in the intermediate pion momentum. Use of a cutoff in the Klein p-wave terms is found to improve substantially the agreement with experiment; however, the calculation is extremely sensitive to the cutoff value chosen. This simply reflects the divergent nature of the critical integrals.

Several alternatives to a sharp momentum cutoff are possible. First, it is apparent that a simple reduction in the power of  $q'$  in the p-wave intermediate integrals will have a beneficial effect. This idea suggests that useful forms for the p-wave scattering Hamiltonians might be

$$H_i^3 = -4\pi\lambda_3\bar{\mu}^3 q^2 \left( \hat{q}_\pi \phi_i \right) \cdot \left( \hat{q}_\pi \phi_i \right)$$

and

$$H_j^4 = -4\pi\lambda_+ \mu^{-3} g^2 \bar{\sigma}_j \cdot \hat{r}_j \cdot \left( \frac{\hat{q}}{g} \phi_j \right) \chi \left( \frac{\hat{q}}{g} \phi_j \right) . \quad (2-30)$$

These forms have the property of kinematical equivalence at low energy to the on-shell free scattering amplitudes. For this reason, we will refer to Eq. (2-30) as the on-shell Hamiltonian. It will be seen that the angular properties of Eq. (2-5) are unchanged; thus the essential replacements in Eq. (2-23) are simply

$$\begin{aligned} \tilde{f}_{31}(r) &\rightarrow \frac{g}{\mu} f_{21}(r), \\ \tilde{f}_{40}(r) &\rightarrow \frac{g}{\mu} f_{30}(r) \end{aligned}$$

and

$$\tilde{f}_{42}(r) \rightarrow \frac{g}{\mu} f_{32}(r) . \quad (2-31)$$

The result using the on-shell p-wave Hamiltonian of Eq. (2-30) is shown by the solid curve in Fig. 4. This curve is seen to reasonably represent the experimental data. This result bears some similarity to that presented by Reitan,<sup>48</sup> who extended his previous calculation to 20 MeV pion kinetic energy with new scattering parameters to obtain better agreement with the data of Rose. Reitan's p-wave scattering Hamiltonians are believed similar to those of Eq. (2-30).<sup>82</sup>

We present one other calculation of interest using the on-shell p-wave Hamiltonian. Following a suggestion by Miller,<sup>83</sup> we delete the

s-wave operator from the direct absorption term,  $H^0$ . Certain integrals in Eq. (2-23) are therefore omitted and kinematical corrections are made to those remaining. The calculation is now frame dependent; the result shown in Fig. 4 is obtained in the overall-center-of-mass frame. Again, the small difference between this result and that preceding it reflects the cancellation of the direct s-wave terms. However, the main point is that the calculated cross section for this reaction is relatively insensitive to the presence of the Galilean-invariant term.

Although the p-wave Hamiltonian terms given in Eq. (2-30) are reasonably successful in reproducing the experimental data, this result would not seem to have a firm theoretical basis. First, in the low-energy limit, these forms do not reduce to Klein's results given in Eq. (2-4). It also seems reasonable to require that the Hamiltonian vanish far off-shell, while for the forms in question, the off-shell and on-shell properties are the same and constant.

We turn now to the p-wave scattering Hamiltonian suggested in Eq. (2-5). These terms were created expressly to satisfy the requirements stated above. In this approach,  $\alpha$  is treated as a free parameter. To permit calculation through the absorption resonance region, the scattering parameters,  $\lambda_i$ , are taken to be complex and are determined according to results presented in Appendix VIII rather than those given in Eq. (2-29).

Figure 5 presents results for calculations performed with the full Hamiltonian, including the p-wave terms of Eq. (2-5). These results were obtained using the Hamada-Johnston potential for initial and final nuclear states, and three different values of the off-shell parameter,  $\alpha$ . As can be seen, the calculation shows great sensitivity to this parameter.

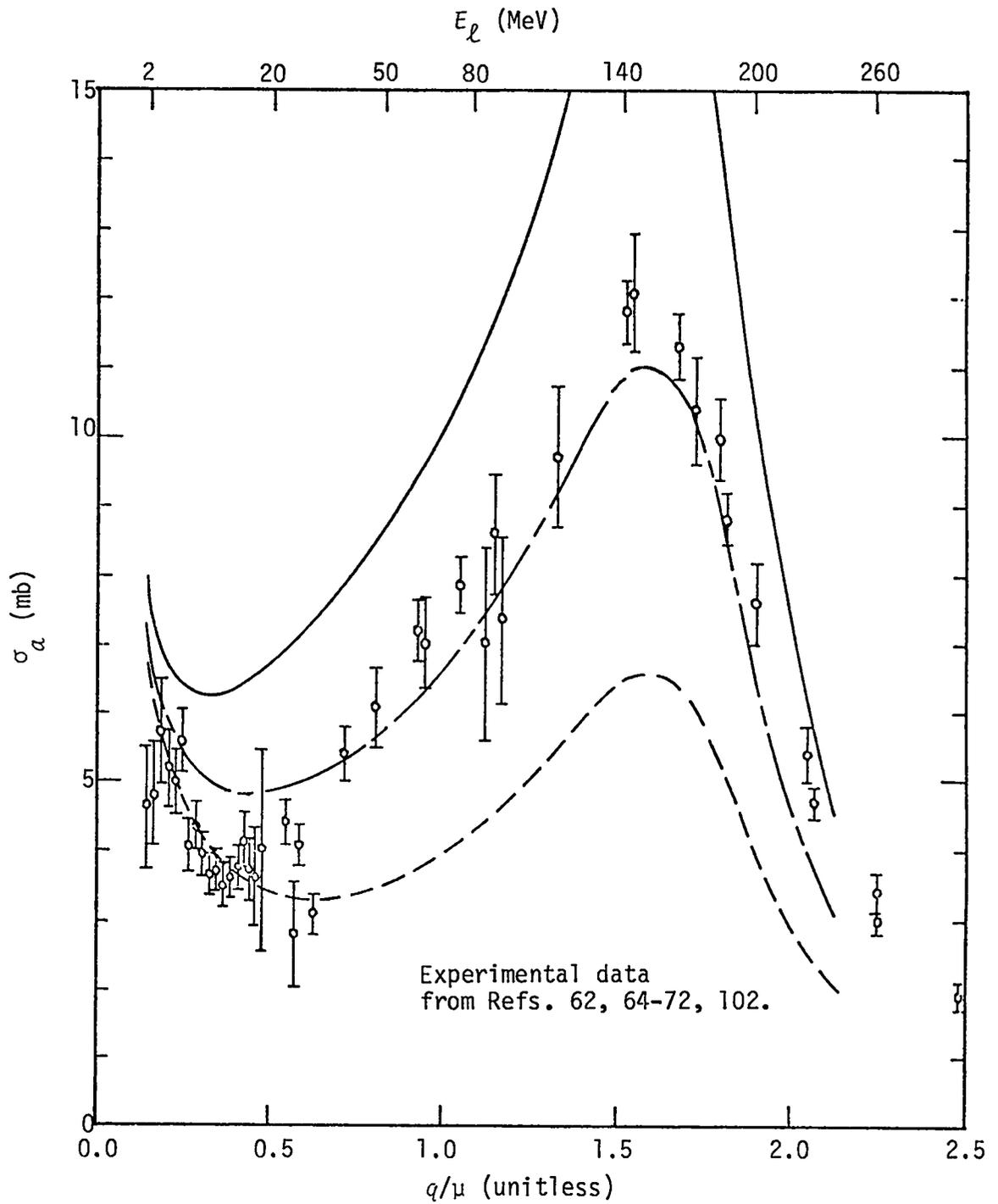


Fig. 5. Cross sections for the separable Hamiltonian and Hamada-Johnston wave functions.

- $\alpha = 200$  MeV/c
- · - · -  $\alpha = 300$  MeV/c
- $\alpha = 400$  MeV/c

A similar set of curves is shown in Fig. 6 for the Boundary Condition Model with 7.55% D-state in the deuteron. These results are seen to be very similar to those obtained with the Hamada-Johnston potential, and exhibit the same degree of sensitivity to the off-shell behavior. The fourth curve in Fig. 6 was obtained using the value,  $\alpha = 300$  MeV/c, and the Boundary Condition Model with 4.60% D-state in the deuteron.

As can be seen from Figs. 5 and 6, a value of  $\alpha$  slightly in excess of 300 MeV/c seems likely to produce a reasonable fit to the experimental data. Previous estimates of this parameter have included 230,<sup>78</sup> 336<sup>77</sup> and 519<sup>76</sup> MeV/c. Thus this calculation would seem extraordinarily sensitive to the off-shell parameter.

It is not difficult to see why this process is so sensitive to the off-shell character of pi-nucleon scattering. If the initial and final states of the nucleons are taken to be on-shell, then the intermediate pion must propagate with an energy,  $q_0/2$  (the external pion must give half its energy to each nucleon). The intermediate momentum is then given by

$$q'^2 = \frac{q^2}{4} - \frac{3\mu^2}{4} . \quad (2-32)$$

Thus knowledge of the behavior of both the absorption and scattering vertices is required far off-shell. It is assumed that Eq. (2-2) represents the absorption vertex correctly, although at higher pion momenta this can be questioned. But the essential point is that the off-shell momentum enters the matrix element through both vertices.

We turn now to angular distributions calculated using the value,  $\alpha = 300$  MeV/c. Results in the following figures present comparisons

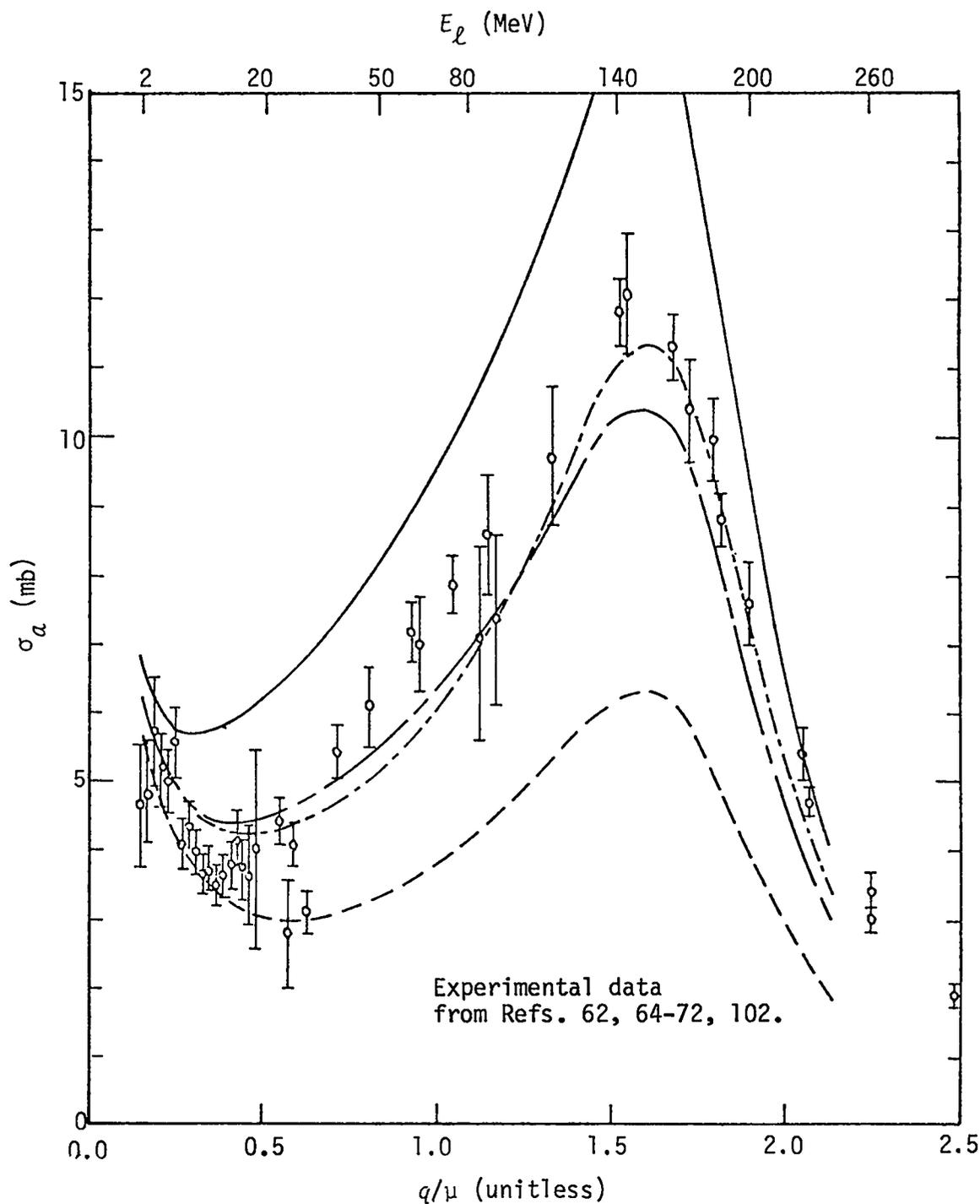


Fig. 6. Cross sections for the separable Hamiltonian and Boundary Condition Model wave functions.

- $\alpha = 200$  MeV/c, 7.55% D-state
- - - - -  $\alpha = 300$  MeV/c, 7.55% D-state
- $\alpha = 400$  MeV/c, 7.55% D-state
- . . . -  $\alpha = 300$  MeV/c, 4.60% D-state

for the three different nucleon-nucleon potentials. For the low-energy data, we follow precedent and plot the ratio of the two leading coefficients of Eq. (2-16) vs incident pion momentum. These results are compared in Fig. 7 with an experimental data compilation by Rosenfeld.<sup>63</sup> We note that only the coefficients,  $C_0$  and  $C_2$ , enter our calculated ratio. On the other hand, the experimentally determined ratios are probably the result of forcing the data to a degree-two fit, and may thus include higher-order terms. Therefore, use of this ratio may be misleading, and we would prefer comparison directly with the cross section measurements.

In Fig. 7, the Boundary Condition Model with 7.55% D-state is seen to give similar results to the Hamada-Johnston, which has a 7.00% D-state. The 4.60% D-state BCM case lies considerably above the other two, thereby implying a possible sensitivity to the fraction of D-state in the deuteron. The previously mentioned experiment at LAMPF is designed for accuracy sufficient, in principle, to permit selection of a preferred nuclear model.

For higher energies, we follow the usual convention of plotting the differential cross section vs  $\cos^2\theta$ . These results are compared with experimental data at six pion kinetic energies in Figs. 8 and 9. The calculated results are seen to give reasonable agreement with experiment, particularly below and near the resonance.

Above the resonance, there is experimental evidence for negative coefficients in the higher powers of  $\cos^2\theta$ . It should be pointed out that this calculation has neglected multiple-scattering graphs as well as the nonunitarity of the proton-proton wave functions. This absorption



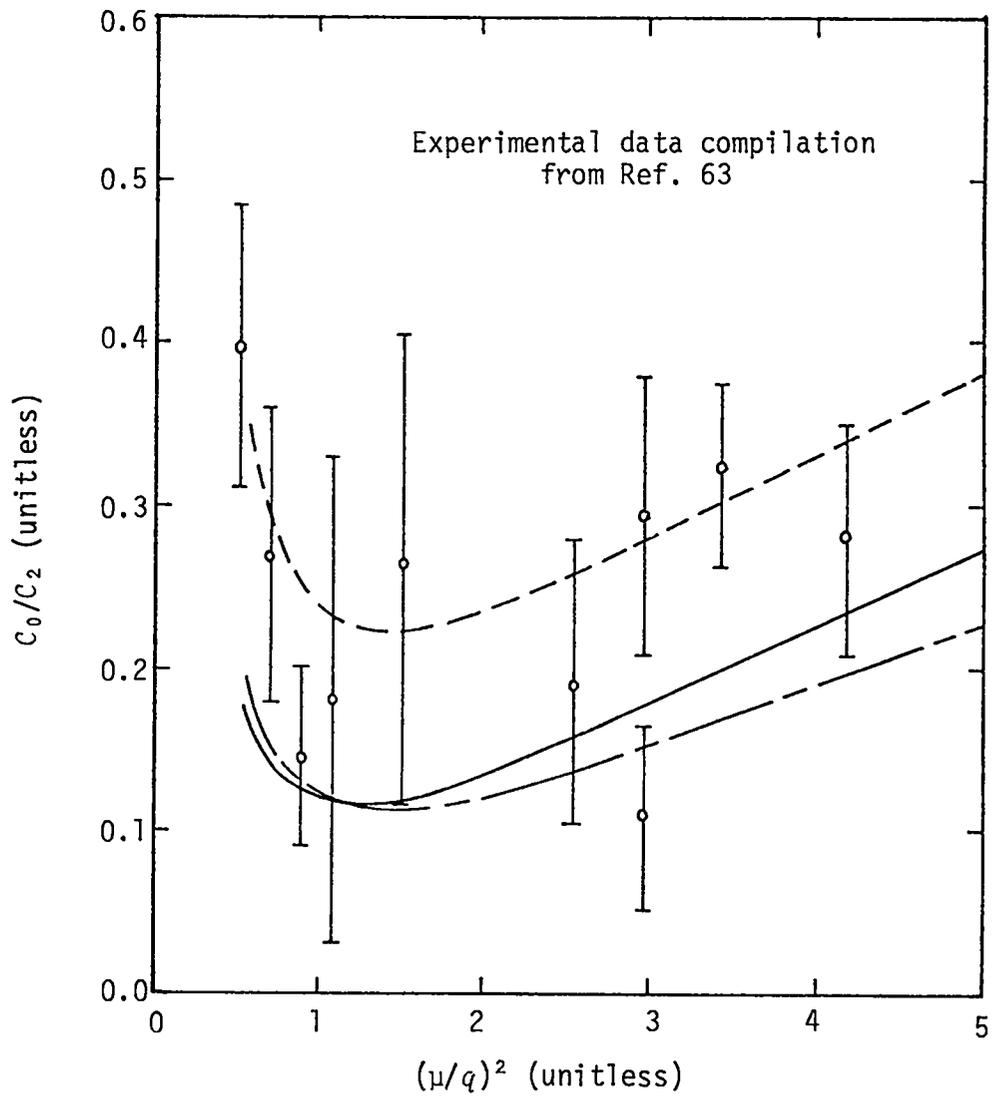


Fig. 7. Differential cross section coefficients for the separable Hamiltonian with  $\alpha = 300$  MeV/c.

- Boundary Condition Model with 4.60% D-state
- . - . - . Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

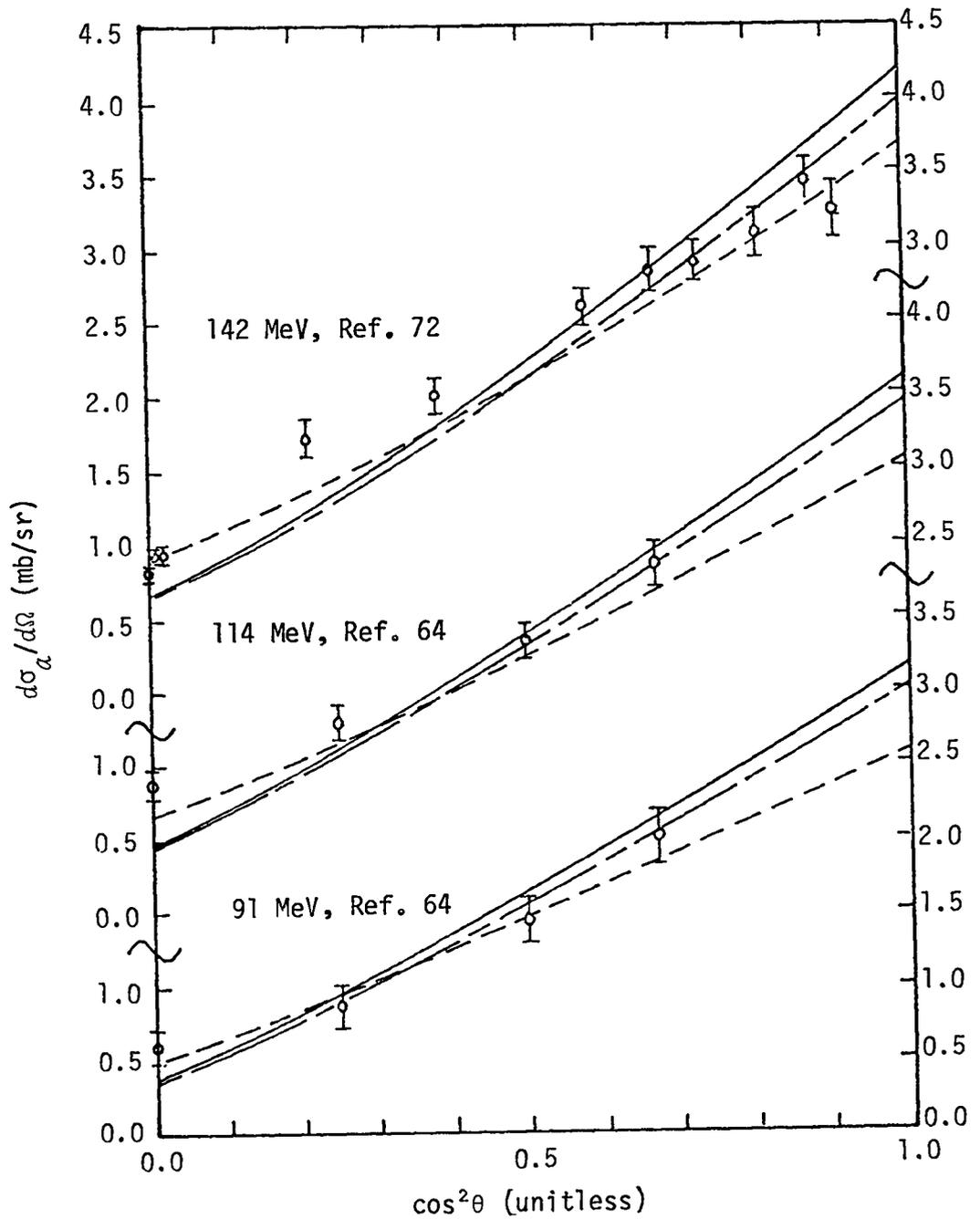


Fig. 8. Differential cross sections at 91, 114 and 142 MeV for the separable Hamiltonian with  $\alpha = 300$  MeV/c.

- Boundary Condition Model with 4.60% D-state
- - - - - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

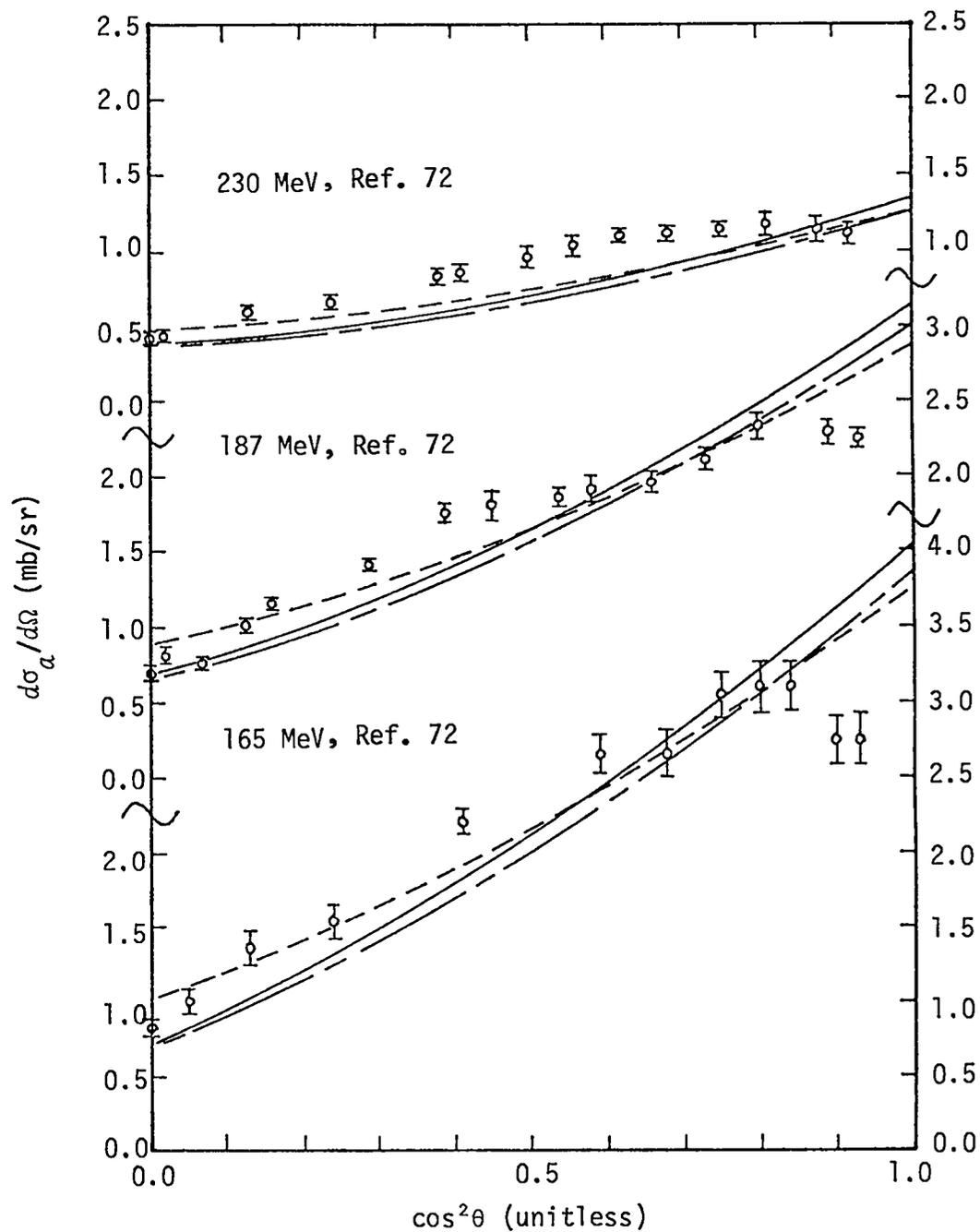


Fig. 9. Differential cross sections at 165, 187 and 230 MeV for the separable Hamiltonian with  $\alpha = 300$  MeV/c.

- Boundary Condition Model with 4.60% D-state
- · - · - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

effect is particularly large above the resonance for the  ${}^3D_1$  state, which is dominant in this energy region.

On the whole, however, this calculation gives a satisfactory representation of the experimental data. Further, the results show sensitivity both to off-shell pi-nucleon scattering and to the nucleon-nucleon potential.

It may be asked why the calculation which uses the on-shell pi-nucleon amplitudes given in Eq. (2-30) works as well as it does. The answer may lie in the following considerations. The on-shell replacements substitute  $q^2 \hat{q} \cdot \hat{q}'$  in an integral which involves other functions of  $q'$ . We have just argued that the proper replacement should be something like  $\bar{q} \cdot \bar{q}' (q^2 + \alpha^2) / (q'^2 + \alpha^2)$ , so that we need to compare the constant,  $q$ , to  $q' (q^2 + \alpha^2) / (q'^2 + \alpha^2)$  over the important range of integration. The magnitudes of these two functions are very similar in this region. However, for the reasons stated previously, we regard the cross section fits obtained using the constant on-shell forms of Eq. (2-30) as fortuitous.

## CHAPTER 3

### S-MATRIX FORMALISM

Lazard, Ballot and Becker<sup>49</sup> (LBB) have recently presented an S-matrix calculation based upon Lagrangian formalism for the absorption of a positive pion by a deuteron. This calculation makes extensive use of an earlier work<sup>84</sup> which develops the formalism for obtaining nuclear reaction amplitudes within the framework of a second-quantized theory. Thus, in the pion absorption problem, LBB are naturally led to a plane-wave representation of the two-proton state. This calculation is referred to as the Born approximation. They also calculate terms for the first two pion partial waves using a modified asymptotic form for the  $^3P_1$ ,  $^1S_0$  and  $^1D_2$  two-proton states. These terms constitute the most significant part of the total matrix element; this calculation is referred to as approximation I. In approximation II the remaining pion partial waves are included, using the Born approximation for the two-proton final state.

In this chapter we recast the S-matrix formalism as a nuclear state operator. This is done to permit use of more physical wave functions as well as the matrix element formalism already developed in Chapter 2. The LBB formalism is briefly reviewed before the equivalent nuclear transition operator is given. Calculated results are then presented for several wave functions and approximations.

In the LBB formalism, the S-matrix element is given in terms of in-out formalism by

$$S_{fi} = \langle P'_1 P'_2 | P_d q \rangle^{in} \quad (3-1)$$

The symbols used in this chapter will generally be defined according to conventions established in Chapter 2. The four-momenta symbols in Eq. (3-1) are understood to contain all isospin and spin variables. The S-matrix element is related to the transition matrix element through the relation,

$$S_{fi} = -2\pi i \delta(P'_1 + P'_2 - P_d - q_0) T_{fi} \quad (3-2)$$

where the transition matrix element is given by

$$T_{fi} = \langle P'_1 P'_2 | J_{P_d}^+ | q \rangle \quad (3-3)$$

In this expression,  $J_{P_d}^+$  is the adjoint current operator of the deuteron; it is expressed as

$$J_{P_d}^+ = \int d\bar{P}_1 d\bar{P}_2 \langle P_1 P_2 | P_d \rangle a_{P_1}^+ J_{P_2}^+ \quad (3-4)$$

Here  $a_{P_1}^+$  and  $J_{P_2}^+$  are single-nucleon production and current operators, respectively. Use is made of Eq. (3-4) and the identity,

$$a_{P_1}^+ = a_{P_1}^{+ \text{out}} - i \int_0^\infty dx_0 J_{P_1}^+(x_0), \quad (3-5)$$

in Eq. (3-3) to gain the result,

$$\begin{aligned} T_{fi} = & \int d\bar{P}_2 \langle P_1' P_2' | P_d \rangle \langle P_2' | J_{P_2}^+ | q \rangle - \int d\bar{P}_1 \langle P_2' P_1' | P_d \rangle \langle P_1' | J_{P_1}^+ | q \rangle \\ & + \int d\bar{P}_1 d\bar{P}_2 \langle P_1' P_2' | P_d \rangle^{\text{out}} \langle P_1' P_2' | J_{P_1}^+ (E - H + i\epsilon)^{-1} J_{P_2}^+ | q \rangle. \end{aligned} \quad (3-6)$$

The first two terms in this equation represent direct absorption processes; these terms may be identified with the first graph shown in Fig. 3 of Chapter 2. The third term in Eq. (3-6) contains all of the scattering effects, and is written explicitly as

$$\begin{aligned} & \int d\bar{P}_1 d\bar{P}_2 d\bar{q}' \langle P_1' P_2' | P_d \rangle \left\{ \frac{\langle P_1' | J_{P_1}^+ | q' \rangle \langle P_2' q' | J_{P_2}^+ | q \rangle^{\text{out}}}{E - P_1' - q_0' - P_2' + i\epsilon} \right. \\ & \left. + \frac{\langle P_2' q' | J_{P_2}^+ | q \rangle^{\text{out}} \langle P_1' | J_{P_1}^+ | q' \rangle}{E - P_2' - q_0' - q_0 - P_1' + i\epsilon} \right\} + \text{exchange } \{1 \leftrightarrow 2\}. \end{aligned} \quad (3-7)$$

It is now possible to identify these terms with the last two graphs shown in Fig. 3. Expressions for the absorption and scattering vertices will be given shortly.

In the LBB formalism, the deuteron vertex is given by

$$\langle P_1 P_2 | P_d \rangle = \sqrt{2} \delta(\bar{P}_1 + \bar{P}_2 - \bar{P}_d) \begin{bmatrix} 0 & 0 \\ 1/2 \mu_1 & \\ 1/2 \mu_2 & \end{bmatrix} \langle 1/2 M_1, 1/2 M_2 | \phi(\frac{\bar{P}_1 - \bar{P}_2}{2}) | 1 M \rangle, \quad (3-8)$$

where  $\phi$  is the Fourier transform of the deuteron wave function:

$$\phi(\frac{\bar{P}_1 - \bar{P}_2}{2}) = (2\pi)^{-3/2} \int d\bar{r} e^{-i \frac{\bar{P}_2 - \bar{P}_1}{2} \cdot \bar{r}} \frac{1}{\sqrt{4\pi}} \left\{ \frac{u}{r} + \frac{1}{\sqrt{8}} \frac{w}{r} S_{12}(\hat{r}) \right\}. \quad (3-9)$$

In this equation,  $u$  and  $w$  represent the S- and D-states of the deuteron, respectively. The tensor operator,  $S_{12}(\hat{r})$ , is defined in Appendix II.

LBB assume that the absorption vertex is given by the nonrelativistic reduction of the pseudovector interaction, or

$$\langle P'_1 | J_{P'_1}^+ | q \rangle = i \frac{f}{\mu} (2\pi)^{-3/2} \delta(\bar{P}'_1 - \bar{P}_1 - \bar{q}) \sqrt{\frac{1}{2g_0}} \langle 1/2 M'_1 | \bar{\sigma}_1 \cdot \bar{Q}_{qP'_1} | 1/2 M_1 \rangle f(\mu'_1, \mu_1), \quad (3-10)$$

where

$$\bar{Q}_{qP'_1} = \left( 1 \pm \frac{g_0}{2M} \right) \bar{q} \mp \frac{g_0}{M} \bar{P}'_1 \quad (3-11)$$



for the forward (backward) propagating pion, and

$$f(\mu', \mu) = \begin{cases} \sqrt{2} \delta_{\mu', 1/2} \delta_{\mu, -1/2} \\ \delta_{\mu', \mu} (\delta_{\mu, 1/2} - \delta_{\mu, -1/2}) \\ \sqrt{2} \delta_{\mu', -1/2} \delta_{\mu, 1/2} \end{cases} \quad (3-12)$$

for the absorption of positive, neutral and negative pions, respectively.

They choose for the scattering vertex the form,

$$\begin{aligned} & \langle P'_1 \bar{q}' | J_{P'_1}^+ | q \rangle = \\ & - \frac{\delta(\bar{P}'_1 + \bar{q}' - \bar{P}_1 - \bar{q}) \tilde{W}}{(2\pi)^2 \sqrt{B'_0 B'_0 P'_0 P_0}} \langle 1/2 M'_1 | f_1 + \bar{\sigma}_1 \cdot \hat{q}' \bar{\sigma}_2 \cdot \hat{q} f_2 | 1/2 M_1 \rangle. \end{aligned} \quad (3-13)$$

In this expression,  $\tilde{W}$  is the total pi-nucleon on-shell energy. The partial-wave functions,  $f_1$  and  $f_2$ , are specified in terms of on-shell scattering amplitudes, written symbolically as  $l_{2t, 2j}$ . LBB define for the process,  $\pi^+ + p \rightarrow \pi^+ + p$ ,

$$f_1 = S_{31} + 3 \hat{q}' \cdot \hat{q} P_{33}$$

and

$$f_2 = P_{31} - P_{33}, \quad (3-14)$$

and for the process,  $\pi^+ + n \rightarrow \pi^0 + p$ ,

$$f_1 = \frac{\sqrt{2}}{3} \left\{ (S_{11} - S_{31}) + 3 \hat{q}' \cdot \hat{q} (P_{13} - P_{33}) \right\}$$

and

$$f_2 = \frac{\sqrt{2}}{3} \left\{ P_{11} - P_{31} - P_{13} + P_{33} \right\} . \quad (3-15)$$

In Appendix VIII, the process amplitudes are found to differ in sign from those given in Eq. (3-15); however, inspection of the isospin operators of Eq. (2-7) indicates that terms for this process and the one preceding it are to be subtracted, whereas LBB add terms to obtain the absorption matrix element.

The essential feature of this calculation is now apparent from inspection of Eqs. (3-13), (3-14) and (3-15). That is, LBB assume that the off-shell behavior of the scattering amplitudes is identical to their on-shell behavior. This calculation is thus similar in principle to that performed in Chapter 2 using the p-wave Hamiltonian of Eq. (2-30).

Finally, LBB write the differential cross section as

$$\frac{d\sigma_a}{d\Omega} = (2\pi)^{10} \frac{M q_0}{2} \frac{k'}{q} \frac{1}{3} \sum_{spins} |M_{fi}|^2 , \quad (3-16)$$

where

$$T_{fi} = (2\pi)^3 \delta(\vec{P}'_1 + \vec{P}'_2 - \vec{P}_d - \vec{q}) M_{fi} . \quad (3-17)$$

$M_{fi}$  is the matrix element calculated in this formalism.

In reconstituting this matrix element as a nuclear operator, we delete the internal sums over spins and the integral over the coordinate,  $r$ . The isospin coupling coefficients and  $f(\mu_1 \mu_1)$  functions are evaluated explicitly. Finally, the plane wave factor is identified as the complex conjugate of the final-state wave function; it, and the deuteron wave function are removed. The resulting operator is renormalized to allow use of Eq. (2-14), which is kinematically equivalent to Eq. (3-16). All expressions for cross section developed in Chapter 2 are then applicable. The nuclear transition operator derived from the S-matrix formalism is given by

$$\begin{aligned}
T &= -i(4\pi)^{1/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \\
&\left[ \left( e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 - e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \left( 1 + \frac{g_2}{2M} \right) \bar{q} - \left( e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 + e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \frac{g_0 \bar{k}}{M} \right. \\
&+ \frac{1}{3(2\pi)^2} G_{15} \left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 + e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \cdot \\
&\quad \left. \frac{\int d\bar{q}' \tilde{W} e^{i\frac{\bar{q}'\cdot\bar{r}}{2}} \left\{ \left( 2 + \frac{F'-F}{g_0} + \frac{F'+F}{2M} \right) \bar{q}' - \frac{F'+F}{M} \bar{k}' \right\}}{\sqrt{k_0 p_0} (F-g_0')(F+g_0')} \right. \\
&+ \frac{1}{3(2\pi)^2} (G_{1P} + G_2) \left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 - e^{i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \cdot \\
&\quad \left. \frac{\int d\bar{q}' \tilde{W} e^{i\frac{\bar{q}'\cdot\bar{r}}{2}} \left\{ \left( 2 + \frac{F'-F}{g_0} + \frac{F'+F}{2M} \right) \bar{q}' - \frac{F'+F}{2M} \bar{k}' \right\} \hat{q}' \cdot \hat{q}}{\sqrt{k_0 p_0} (F-g_0')(F+g_0')} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3(2\pi)^2} G_2 \left\{ \frac{\int d\vec{q}' \tilde{W} e^{i\vec{q}' \cdot \vec{r}} e^{-i\vec{q}' \cdot \vec{r}} \sigma_1 \cdot \left\{ \left( 2 + \frac{F-F'}{q_0'} + \frac{F'+F}{2M} \right) \vec{q}' - \frac{F'+F}{M} \vec{k}' \right\} i \vec{\sigma}_2 \cdot \hat{q}' \times \hat{q}}{\sqrt{k_0' P_0} (F - q_0') (F + q_0')} \right. \\
& \left. - \frac{\int d\vec{q}' \tilde{W} e^{i\vec{q}' \cdot \vec{r}} e^{i\vec{q}' \cdot \vec{r}} \sigma_2 \cdot \left\{ \left( 2 + \frac{F-F'}{q_0'} + \frac{F'+F}{2M} \right) \vec{q}' - \frac{F'+F}{M} \vec{k}' \right\} i \vec{\sigma}_1 \cdot \hat{q}' \times \hat{q}}{\sqrt{k_0' P_0} (F - q_0') (F + q_0')} \right\}
\end{aligned}$$

(3-18)

In writing this expression, use has been made of an alternate form for expressing pi-nucleon scattering amplitudes; correspondence with the form given by LBB is obtained through the relation,<sup>85</sup>

$$\vec{\sigma} \cdot \vec{q}' \vec{\sigma} \cdot \vec{q} = \vec{q}' \cdot \vec{q} + i \vec{\sigma} \cdot \vec{q}' \times \vec{q} . \quad (3-19)$$

We have also used the LBB symbols for various combinations of the free-scattering amplitudes,

$$G_{1S} = 4S_{31} - S_{11} ,$$

$$G_{1P} = 12P_{33} - 3P_{13}$$

and

$$G_2 = 4P_{31} - P_{11} - 4P_{33} + P_{13} , \quad (3-20)$$

as well as the propagator terms,

$$F = P_{00} + q_0 - P_{10} - P_{20}' = P_{20}' - P_{10}$$

and

$$F' = -P_{00} + P_{20} + P_{10}' . \quad (3-21)$$

When evaluated on-shell, both the propagator terms in Eq. (3-21) are about equal to the quantity,  $q_0/2$ , used in the pole model calculation of Chapter 2.

It is possible to make an identification of the operators in Eq. (3-18) with those given in Eq. (2-7). The direct absorption terms, for example, are seen to be very nearly identical.

Following the procedure developed in Appendix VI, we write the matrix element as

$$M = \sum_{\lambda=|M'-M|} C(S'M'M\lambda) Y_{\lambda}^{*M'-M}(\hat{k}, \hat{q}) . \quad (3-22)$$

Again, the complex coefficients in Eq. (3-22) are calculated by summing over a product of angular integrals,  $R_i$ , and one-dimensional radial integrals,  $I_i$ , according to

$$C(S'M'M\lambda) = i^{\lambda+1} (4\pi)^{3/2} f \mu^{-1} \sqrt{\frac{1}{2q_0}} \sum_{J'L'L} \sum_{i=1}^{24} R_i I_i . \quad (3-23)$$

As stated previously, the angular and radial integrals are functions of the pion partial wave as well as initial- and final-state quantum numbers which must be summed over in calculating coefficients. Again, the contributing states are identified in Table I of Chapter 1.

Expressions for the angular integrals,  $R_{\lambda}$ , are given explicitly in Appendix VI. The radial integrals for this formalism are now defined as follows:

$$\begin{aligned}
I_1 &= -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} - \frac{\lambda}{r}\right) j_{\lambda}, \\
I_2 &= -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} + \frac{\lambda+1}{r}\right) j_{\lambda}, \\
I_3 &= -\frac{g_0}{M} \int dr r^2 \frac{u}{r} j_{\lambda} \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*, \\
I_4 &= -\frac{g_0}{M} \int dr r^2 \frac{u}{r} j_{\lambda} \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \\
I_5 &= -\mu \frac{1}{6} \mu G_{15} \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} f_{31} j_{\lambda} g'^*, \\
I_6 &= -\mu \frac{1}{6} \mu G_{15} \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} f_{31} j_{\lambda} g'^*, \\
I_7 &= +\frac{1}{6} \mu G_{15} \frac{g_0}{M} \int dr r^2 \frac{u}{r} f_{20} j_{\lambda} \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*, \\
I_8 &= +\frac{1}{6} \mu G_{15} \frac{g_0}{M} \int dr r^2 \frac{u}{r} f_{20} j_{\lambda} \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \\
I_9 &= +\mu \frac{1}{6} \mu (G_{1P} + G_2) \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{30} j_{\lambda-1} g'^*, \\
I_{10} &= +\mu \frac{1}{6} \mu (G_{1P} + G_2) \left(2 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{30} j_{\lambda+1} g'^*,
\end{aligned}$$

$$\begin{aligned}
I_{11} &= +\mu \frac{1}{6} \mu (G_{1P} + G_2) \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{32}^{\sim} j_{\lambda-1}^i g'^*, \\
I_{12} &= +\mu \frac{1}{6} \mu (G_{1P} + G_2) \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{32}^{\sim} j_{\lambda+1}^i g'^*, \\
I_{13} &= -\frac{1}{6} \mu (G_{1P} + G_2) \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda-1}^i \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*, \\
I_{14} &= -\frac{1}{6} \mu (G_{1P} + G_2) \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda-1}^i \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \\
I_{15} &= -\frac{1}{6} \mu (G_{1P} + G_2) \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda+1}^i \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*, \\
I_{16} &= -\frac{1}{6} \mu (G_{1P} + G_2) \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda+1}^i \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \\
I_{17} &= +\mu \frac{1}{6} \mu G_2 \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{30}^{\sim} j_{\lambda-1}^i g'^*, \\
I_{18} &= +\mu \frac{1}{6} \mu G_2 \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{30}^{\sim} j_{\lambda+1}^i g'^*, \\
I_{19} &= +\mu \frac{1}{6} \mu G_2 \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{32}^{\sim} j_{\lambda-1}^i g'^*, \\
I_{20} &= +\mu \frac{1}{6} \mu G_2 \left(2 + \frac{q_0}{2M}\right) \int dr r^2 \frac{u}{r} \tilde{f}_{32}^{\sim} j_{\lambda+1}^i g'^*, \\
I_{21} &= -\frac{1}{6} \mu G_2 \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda-1}^i \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*, \\
I_{22} &= -\frac{1}{6} \mu G_2 \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda-1}^i \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \\
I_{23} &= -\frac{1}{6} \mu G_2 \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda+1}^i \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*
\end{aligned}$$

and

$$I_{24} = -\frac{1}{6} \mu G_2 \frac{q_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{21}^{\sim} j_{\lambda+1}^i \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*. \quad (3-24)$$

These results may be compared with those in Eq. (2-23).

The s- and p-wave intermediate integrals are now given by

$$f_{20}(r) = \left(\frac{q_0}{M}\right)^{-1} \frac{2}{\pi} \mu^{-1} \int_0^{\infty} \frac{dq' q'^2 j_0(q'r)}{(F-q'_0)(F+q'_0)} \left(\frac{F'+F}{M}\right) \frac{\tilde{W}}{\sqrt{k'_0 P_0}},$$

$$f_{31}(r) = \left(2 + \frac{q_0}{2M}\right)^{-1} \frac{2}{\pi} \mu^{-2} \int_0^{\infty} \frac{dq' q'^3 j_1(q'r)}{(F-q'_0)(F+q'_0)} \left(2 + \frac{F'-F}{q'_0} + \frac{F'+F}{2M}\right) \frac{\tilde{W}}{\sqrt{k'_0 P_0}},$$

$$\tilde{f}_{21}(r) = \left(\frac{q_0}{M}\right)^{-1} \frac{2}{\pi} \mu^{-1} \int_0^{\infty} \frac{dq' q'^2 j_1(q'r)}{(F-q'_0)(F+q'_0)} \left(\frac{F'+F}{M}\right) \frac{\tilde{W}}{\sqrt{k'_0 P_0}},$$

$$\tilde{f}_{30}(r) = \left(2 + \frac{q_0}{2M}\right)^{-1} \frac{2}{\pi} \mu^{-2} \int_0^{\infty} \frac{dq' q'^3 j_0(q'r)}{(F-q'_0)(F+q'_0)} \left(2 + \frac{F'-F}{q'_0} + \frac{F'+F}{2M}\right) \frac{\tilde{W}}{\sqrt{k'_0 P_0}}$$

and

$$\tilde{f}_{32}(r) = \left(2 + \frac{q_0}{2M}\right)^{-1} \frac{2}{\pi} \mu^{-2} \int_0^{\infty} \frac{dq' q'^3 j_2(q'r)}{(F-q'_0)(F+q'_0)} \left(2 + \frac{F'-F}{q'_0} + \frac{F'+F}{2M}\right) \frac{\tilde{W}}{\sqrt{k'_0 P_0}}.$$

(3-25)

This choice of normalization is made to allow comparison with similar functions in Chapter 2. The important difference in form, of course, arises from the treatment of the p-wave scattering. However, the forms given above correspond closely to those resulting from the on-shell Hamiltonian of Eq. (2-30). In the present case, the intermediate



integrals are performed numerically; in absence of a cutoff, use is made of the recursion relations,

$$f_{n+1, \ell-1}(r) = \tilde{\mu}^{-1} \left( \frac{d}{dr} + \frac{\ell+1}{r} \right) f_{n, \ell}(r)$$

and

$$f_{n+1, \ell+1}(r) = -\tilde{\mu}^{-1} \left( \frac{d}{dr} - \frac{\ell}{r} \right) f_{n, \ell}(r). \quad (3-26)$$

Several approximations for the propagator are possible in evaluating these integrals, including a pole model calculation similar to that used in Chapter 2, the calculations of LBB in which the Fermi motion is taken to be zero, and, at the other extreme, a calculation in which the initial nucleon energy is determined by momentum conservation at the scattering vertex. These differences are found, however, to have only small effect on the calculated cross section.

The full calculation outlined above includes all pion partial waves and thus corresponds closely with LBB's approximation II. We have modified their formalism to include for all pion partial waves the final-state two-proton interaction as determined from solution of the Schrödinger equation.

In addition to the above calculation, we have also coded approximation I of LBB. This calculation is given explicitly in their paper in terms of integrals, matrix elements and angular dependences. Approximation I includes only the first two pion partial waves; specifically, twenty terms involving the  ${}^3P_1$ ,  ${}^1S_0$  and  ${}^1D_2$  states are given. To check

the formalism developed in Chapter 2 and used in similar form here (beginning with Eq. (3-18)), comparisons were made directly with approximation I calculations. In approximation I, the external pion wave function is set to one. Thus to obtain agreement between the two calculations it is necessary to modify the spherical Bessel functions in the radial integrals in Eq. (3-24) in addition to removing about one hundred terms from the matrix element (for the first two partial waves alone). However, the two explicit calculations can be shown equivalent and are found to give numerical agreement. Agreement is also obtained with the original approximation I calculations performed by Lazard.<sup>86</sup> For this verification work, use was made of the deuteron pole models of Gourdin<sup>87</sup> and McGee<sup>88</sup> and the final-state wave function proposed by LBB. This wave function is given by

$$\psi^*(r) = e^{-i\delta_{L'}} \kappa_{L'} \left\{ \cos \delta_{L'} j_{L'}(kr) - f_{L'} \sin \delta_{L'} \eta_{L'}(kr) \right\}, \quad (3-27)$$

where  $\kappa_{L'}$  is the scattering absorption coefficient, and the cutoff factor multiplying the Neumann function is

$$f_{L'} = (1 - e^{-zr})^{L'+1}. \quad (3-28)$$

In this expression  $z$  is the parameter used by LBB in their absorption calculations to explore sensitivity to the short-range interaction. For our comparisons, the phase shifts used are MacGregor's solution 4.<sup>89</sup>

With regard to calculations using approximation I, we make the following comments. First, we find that deletion of the (pion) spherical Bessel functions from the radial integrals has the effect of increasing the total cross section by about 10%. Use of the phase factor,  $e^{-i\delta_L'}$ , in the wave function, while not in agreement with results obtained in Appendix V, can be shown not to affect this calculation. With use of Eq. (3-27), we find the cross section at high energy to be dominated by the effect of nonunitarity ( $\alpha_L < 1$ ) in the  $^1D_2$  state. As also noted by LBB, the Born calculation gives results some 30% in excess of approximation I at the resonance. If the nonunitarity of the  $^1D_2$  is neglected (the  $^3P_1$  is not important at high energy) so that only the real part of the MacGregor phase shift is used in Eq. (3-27), the effect is actually to enhance the calculated cross section over the Born term. On the other hand, at low energy the nuclear scattering matrix is unitary, and inclusion of the  $^3P_1$  phase shifts lowers the calculated values from the Born result to near the experimental values. We also find that the shift in the resonance seen in going from approximation I to approximation II comes not from the additional pion partial waves, but the inclusion of operator terms which are omitted in approximation I. The effect of adding higher partial waves seems to be an overall enhancement of the cross section as opposed to a shift in resonance. These features will be seen to carry over into calculations using the Boundary Condition Model and Hamada-Johnston wave functions. We now present the results of several calculations using these wave functions.

Figure 10 shows results obtained for approximation I. The spatial dependence of the external pion wave function has been included in the calculation, however, so that approximation I here refers only to a

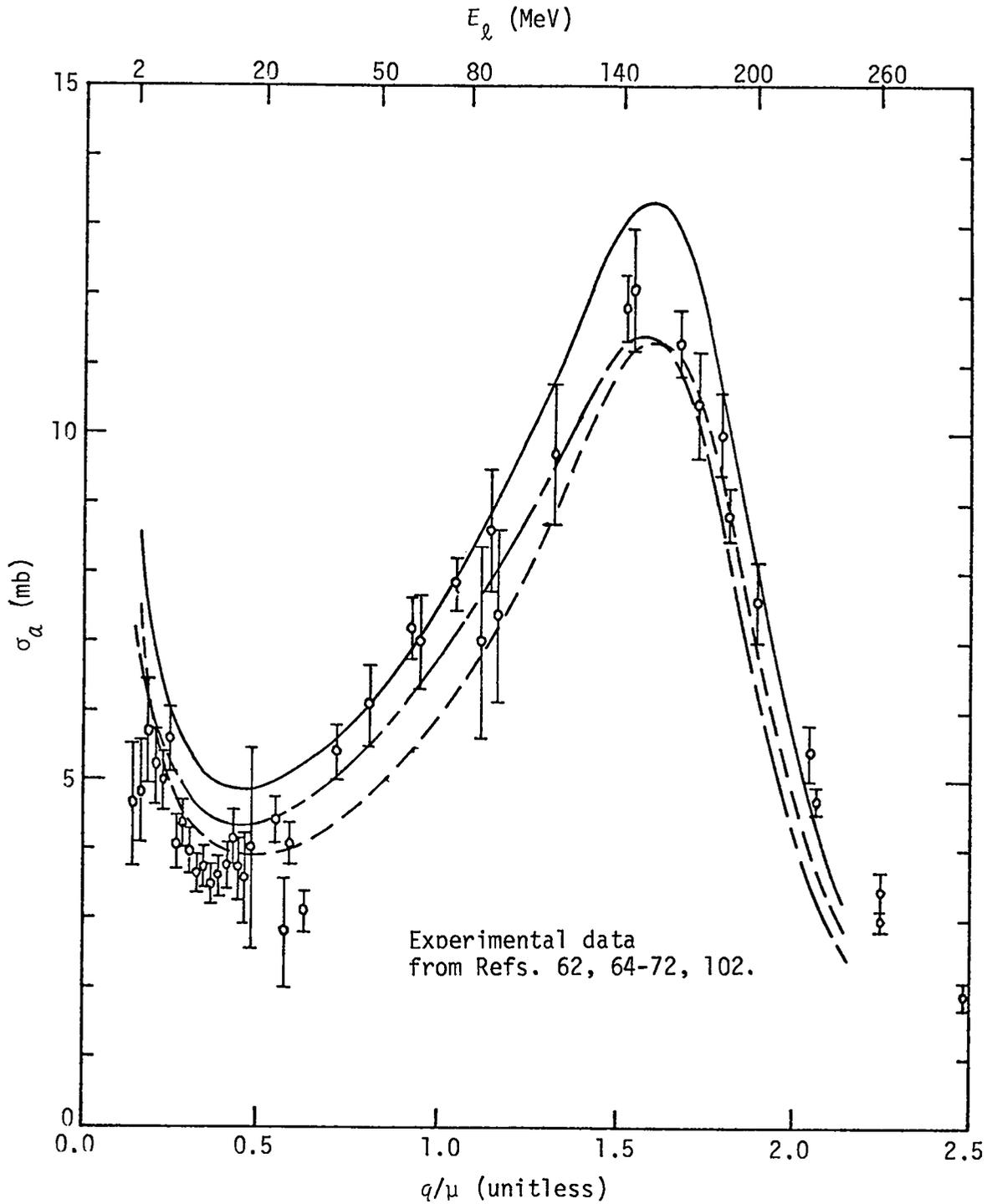


Fig. 10. Cross sections for LBB approximation I.

- Boundary Condition Model with 4.60% D-state
- · - · - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

specific selection of the terms given in Eq. (3-23). The pi-nucleon phase shifts used are those of McKinley;<sup>81</sup> amplitudes are evaluated in the two-body center-of-mass frame. The propagator is calculated assuming momentum conservation at the scattering vertex; however, as also noted by LBB, this makes little difference in the calculation. It does have the effect of making the intermediate integrals converge more rapidly. Figure 10 illustrates an effect also noted in Chapter 2: that is, the percentage of D-state in the deuteron is found to have surprisingly little effect on total cross section. This possibly may be explained in terms of the quadrupole moment, the equation for which has been expressed in Appendix IV. Since the first term in Eq. (IV-6) dominates, an ad hoc renormalization of the D-state wave function, while presumably having greater effect upon the absorption calculation, will detrimentally affect the quadrupole moment, which is a relatively well-known experimental quantity. This illustrates the importance of varying the percentage of D-state in a consistent way by changing potential or boundary terms used in solving the Schrödinger equation. The latter course is followed for the Boundary Condition Model.

A second observation involves the extent of agreement with experiment beyond the resonance. The wave functions used to obtain the results shown in Fig. 10 are, of course, unitary. This behavior thus differs markedly from that previously described involving the use of Eq. (3-27).

The effect of possible uncertainties in the pi-nucleon phase shifts is explored by also using the CERN 1 theoretical-fit data.<sup>90</sup> For this calculation, the numerical data are spline-fitted in momentum to allow interpolation between points. Below 10 MeV, forms of first and third degree appropriate to the phase shift are used; otherwise the form is

taken to be a degree-three polynomial. Results for this calculation are compared with those using the McKinley parameterization in Fig. 11. In both cases the Boundary Condition Model with 7.55% D-state is used. The CERN phase shifts appear to slightly improve the fit for approximation I; however, the difference does not seem significant. This conclusion may be generalized to include all results obtained in Chapters 2 and 3. The calculation in Chapter 4 shows more sensitivity to the phase shifts due to their use in a pion optical model.

Figure 12 presents results for the full calculation using all terms in the expression (3-23) and all pion partial waves. In practice, it was found that the fifth partial wave had insignificant effect upon either total cross section or angular distribution; thus calculations were usually terminated with the fourth partial wave,  $\lambda = 3$ . The general effect previously noted for the inclusion of missing terms and higher partial waves can be seen by comparing these results with those in Fig. 10. The full calculation does not appear to compare favorably with approximation I in agreement with experimental data.

The next three figures present differential cross sections obtained for the full calculation. Following the convention established in Chapter 2, we calculate the ratio of the two leading coefficients in Eq. (2-16) to illustrate angular dependence at low energy. These results are shown in Fig. 13. Comparison may be made with results shown in Fig. 7 for the off-shell Hamiltonian. We find a similarity in these results, particularly regarding the apparent effect of the D-state fraction in the deuteron.

Angular distributions at six higher energies are shown in Figs. 14 and 15, which make evident the necessity for considering higher pion partial waves. It is found that inclusion of the third partial wave

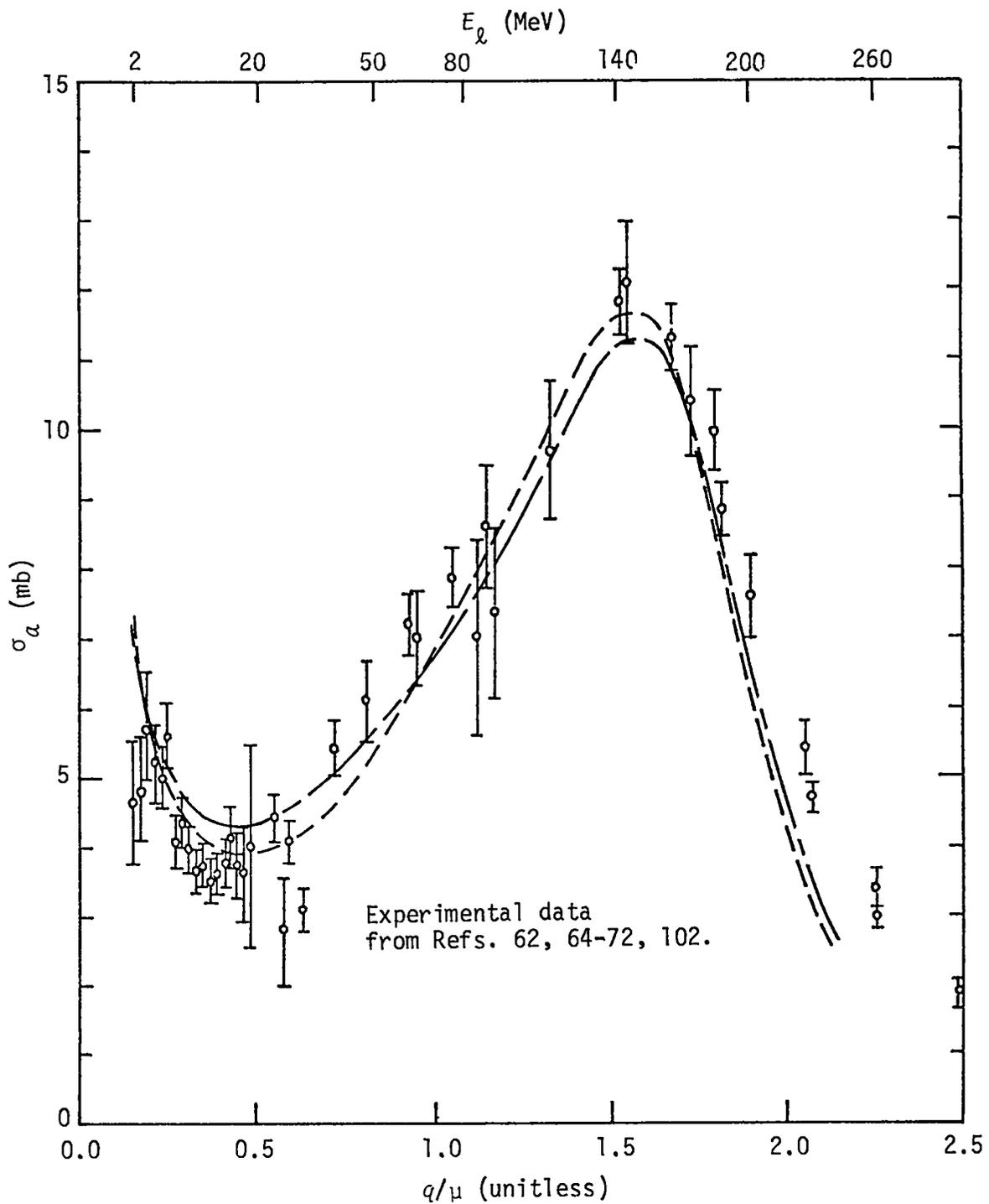


Fig. 11. Cross sections using two different pi-nucleon phase shift parameterizations.

- CERN 1 theoretical-fit phase shifts
- McKinley phase shifts

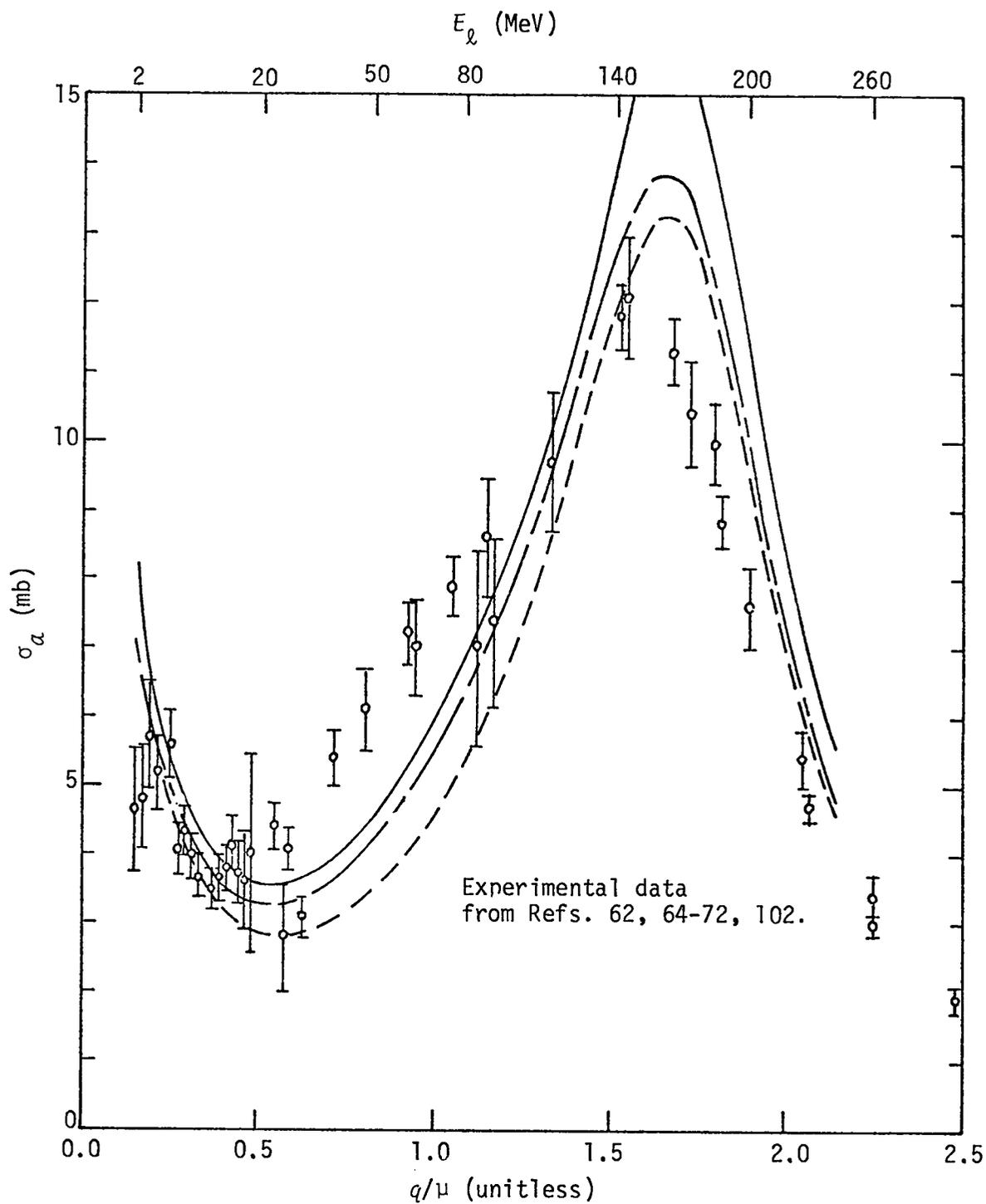


Fig. 12. Cross sections for LBB formalism with all pion partial waves.

- Boundary Condition Model with 4.60% D-state
- · - · - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state



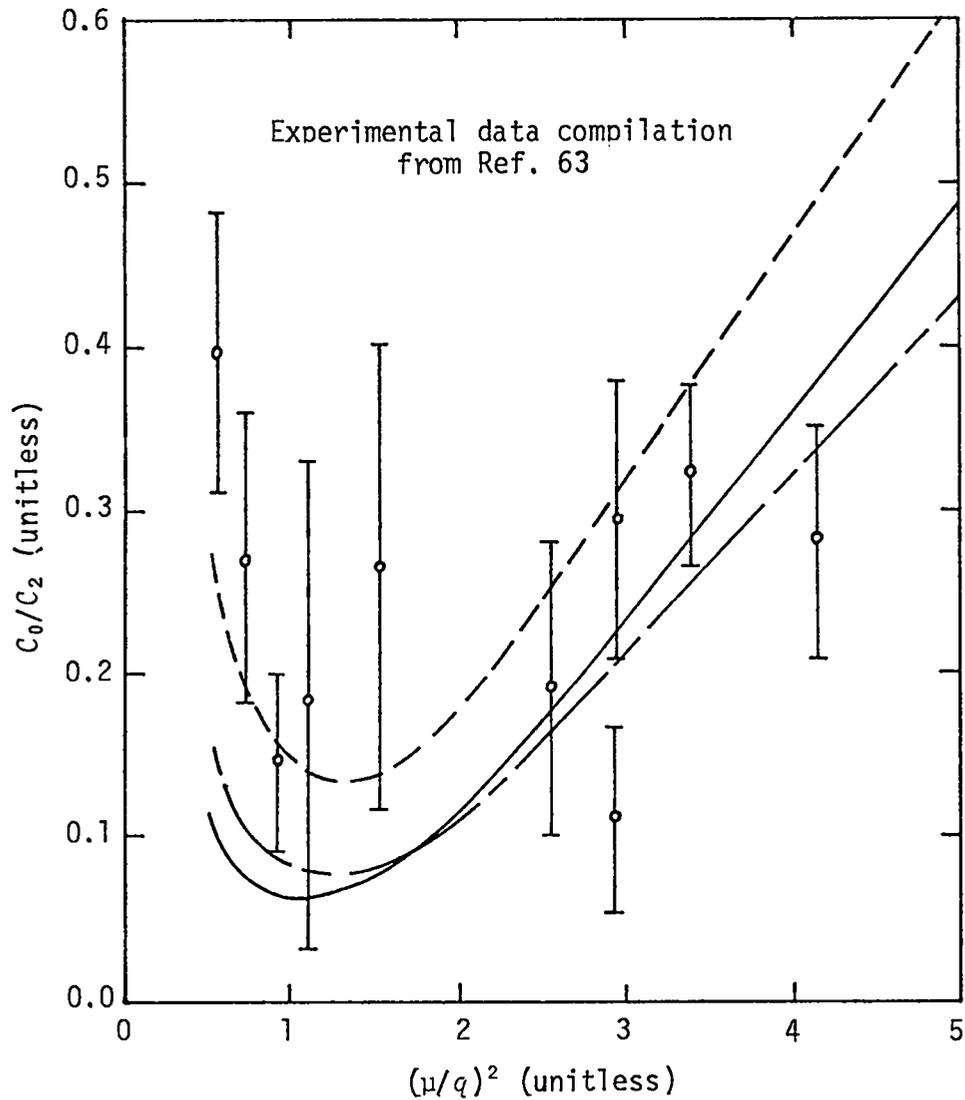


Fig. 13. Differential cross section coefficients for LBB formalism with all pion partial waves.

- Boundary Condition Model with 4.60% D-state
- · - · - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

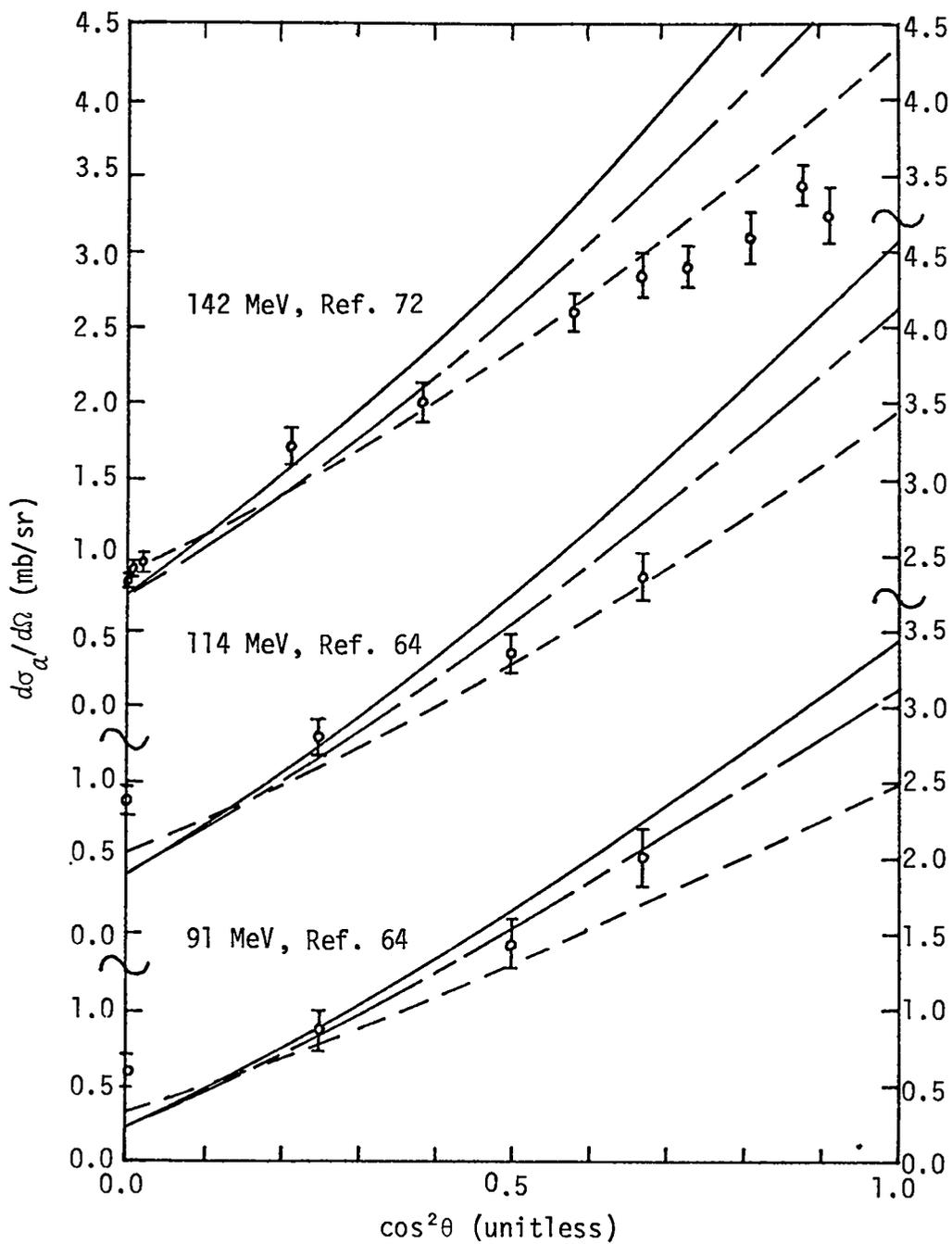


Fig. 14. Differential cross sections at 91, 114 and 142 MeV for LBB formalism with all partial waves.

- Boundary Condition Model with 4.60% D-state
- - - - - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

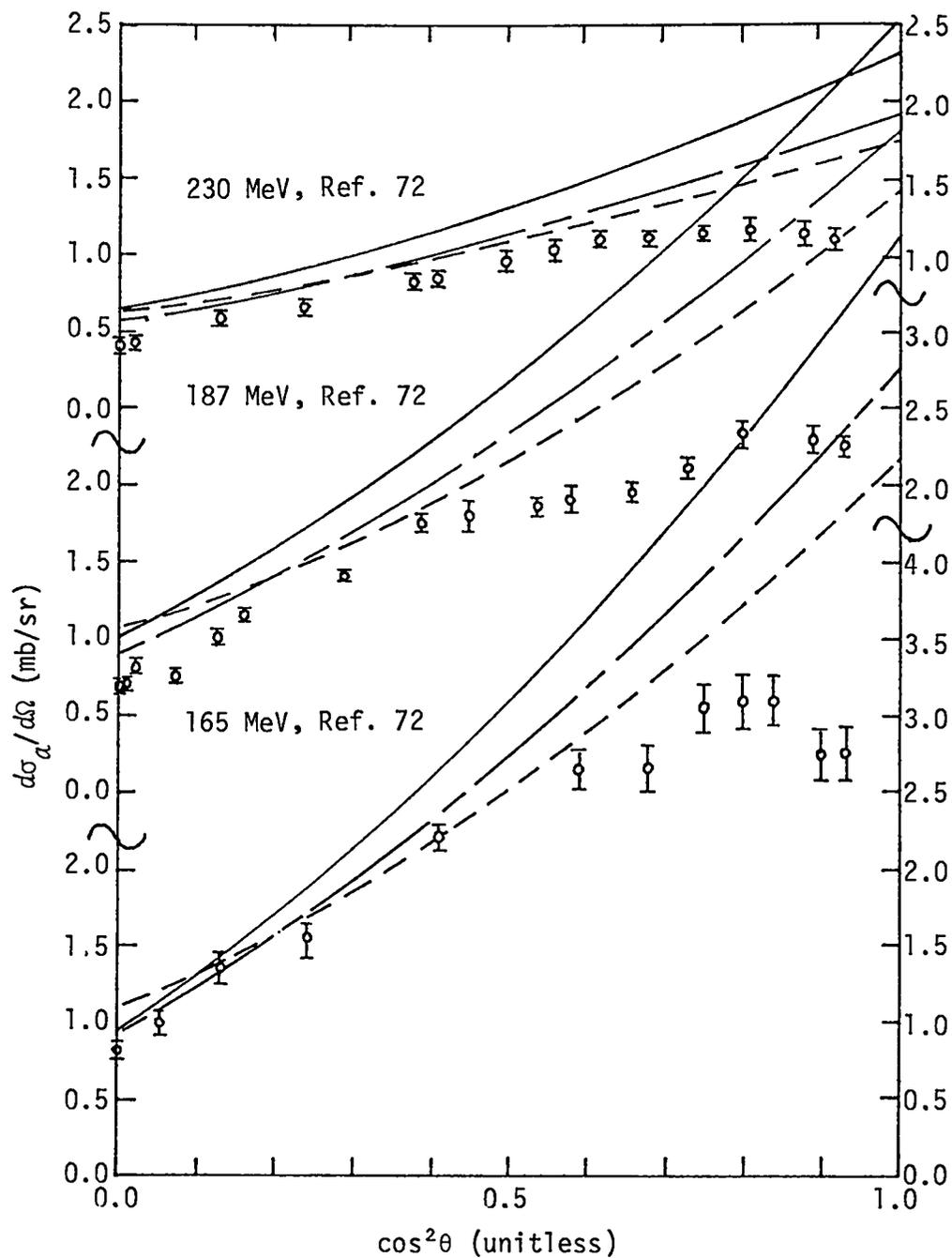


Fig. 15. Differential cross sections at 165, 187 and 230 MeV for LBB formalism with all partial waves.

- Boundary Condition Model with 4.60% D-state
- - - - - Boundary Condition Model with 7.55% D-state
- Hamada-Johnston potential with 7.00% D-state

does, indeed, provide a negative  $\cos^4\theta$  coefficient at higher energies, thus better fitting the experimental data. However, when the fourth partial wave is added, the coefficient becomes overwhelmingly positive, and this is illustrated clearly in Fig. 15. Thus the angular distributions seem slightly worse in this respect than those given previously by LBB for approximation II.

CHAPTER 4  
OPTICAL MODEL FORMALISM

In this chapter, a new approach to the absorption problem is presented in which scattering graphs are incorporated through a pion optical model. In this calculation, only those terms in the Hamiltonian which are linear in the external pion field are retained. The incident pion plane wave is then modified using a pi-nucleus potential to reflect the scattering effect.

First, the general pion optical model formalism is developed; various forms and kinematical transformations are presented. The deuteron density function is obtained from the nucleon-nucleon S-state wave function, taking into account the pi-nucleon interaction distance as a free parameter. Results for the scattering problem using this formalism are presented. Next, a second-order formalism for the deuteron is developed in anticipation of absorption calculations. Scattering calculations show an improvement over those obtained with the general formalism. Finally, the absorption problem is considered using this second-order scattering formalism; identification is made with previously given graphs. Great sensitivity is shown to various optical model parameters.

We briefly review multiple scattering theory, which provides a fundamental basis for the pion optical model. The total Hamiltonian for the pi-nucleus system is given by

$$H = H_0 + V , \tag{4-1}$$

where the pi-nucleus potential,  $V$ , is assumed to be a sum of two-body (pi-nucleon) interactions:

$$V = \sum_{i=1}^A V_i . \quad (4-2)$$

The other term,  $H_0$ , is written as a sum of two parts,

$$H_0 = H_A + T_\pi , \quad (4-3)$$

where  $H_A$  is the total nuclear Hamiltonian and  $T_\pi$  is the pion kinetic energy operator.

The pion wave function can be separated into plane wave and scattered components:

$$\psi = \phi + \psi_s . \quad (4-4)$$

The Schrödinger equation then can be written as

$$(E - H_A - T_\pi - V)(\phi + \psi_s) = 0 , \quad (4-5)$$

where  $E$  is the total energy of the system. It is understood that Eq. (4-5) operates upon the nuclear wave function. It can then be shown that Eq. (4-5) leads to the result,

$$\psi_s = \bar{a}' \{ V + V \bar{a}' V + \dots \} \phi, \quad (4-6)$$

where the propagator,  $a$ , is given by

$$a = E - H_0 + i\epsilon. \quad (4-7)$$

From Eq. (4-6), it can be seen that the pi-nucleus scattering matrix is a solution of the Lippmann-Schwinger equation:

$$T = \frac{V}{1 - \bar{a}' V}. \quad (4-8)$$

A formal solution for the problem of pi-nucleus scattering is given by the Watson multiple scattering series.<sup>91</sup> This series is of the form,

$$\tilde{T} = \sum_{i=1}^A t'_i + \sum_{\substack{i,j=1 \\ i \neq j}}^A t'_i \bar{a}' t'_j + \dots \quad (4-9)$$

In this expression,  $t'$  is the pion bound-nucleon scattering amplitude. Restrictions on the indices prevent summing successive scatterings on the same nucleon; however, all other combinations are allowed. The matrix element for elastic scattering from the nuclear ground state is then of the form,

$$\langle 0|\tilde{T}|0\rangle = \langle 0|\sum_{i=1}^A t'_i|0\rangle + \sum_{n,m} \langle 0|\sum_{i=1}^A t'_i|n\rangle \langle n|\bar{a}'|m\rangle \langle m|\sum_{\substack{j=1 \\ j \neq i}}^A t'_j|0\rangle + \dots \quad (4-10)$$

In this equation, the propagator matrix element is diagonal. In any case, virtual nuclear states are to be neglected in the coherence approximation, so that Eq. (4-10) becomes

$$\langle 0|\tilde{T}|0\rangle = A\bar{t}' + A(A-1)\bar{t}'\bar{a}'\bar{t}' + \dots \quad (4-11)$$

Use has been made of antisymmetrized nuclear wave functions to obtain a result used above:

$$\langle 0|\sum_{i=1}^A t'_i|0\rangle = A\bar{t}' \quad (4-12)$$

The amplitude,  $\bar{t}'$ , therefore includes a suitable average over isospin.

The impulse approximation consists here of the assumption that the pi-nucleon free scattering amplitudes are not modified in the presence of nuclear matter. That is,

$$t = t' , \quad (4-13)$$



where  $t$  is the free scattering amplitude. The series in Eq. (4-11) can be summed to obtain

$$\langle 0 | \tilde{T} | 0 \rangle = \frac{A \bar{t}}{1 - (A-1) \bar{a}' \bar{t}} . \quad (4-14)$$

Multiplication of this expression by the factor,  $(A-1)/A$ , allows identification with the potential in Eq. (4-8). This result is

$$\langle 0 | V | 0 \rangle = (A-1) \bar{t} . \quad (4-15)$$

For large nuclei ( $A \gg 1$ ), one can make the approximation,

$$\langle 0 | V | 0 \rangle = A \bar{t} . \quad (4-16)$$

This is the usual pion optical model result. In effect, the multiple scattering problem is reduced to a two-body (pi-nucleus) potential problem.

We next develop several common forms of the pion optical model. In momentum space, the two-body potential of Eq. (4-16) is given by

$$\langle \bar{q}' | V | \bar{q} \rangle = A \langle \bar{q}' | A + t \bar{q}' \cdot \bar{q} | \bar{q} \rangle \delta(\bar{q}' - \bar{q}) . \quad (4-17)$$

This expression is seen to explicitly include terms for the first two partial waves of pi-nucleon scattering. Spin-flip terms shown in Appendix VIII may be excluded for elastic scattering from a spin-zero nucleus. An expression for the nuclear form factor is given by

$$S(\bar{q}' - \bar{q}) = \int d\bar{r} e^{i(\bar{q}' - \bar{q}) \cdot \bar{r}} \rho(r), \quad (4-18)$$

where the nuclear density is assumed spherically symmetric and is normalized according to

$$\int d\bar{r} \rho(r) = 1. \quad (4-19)$$

We ultimately desire the potential to operate on a coordinate space wave function, and hence need the result

$$V(\bar{r}) \psi(\bar{r}) = (2\pi)^{-3} A \int d\bar{q} d\bar{q}' d\bar{r}'' d\bar{r}' \\ (1 + t \bar{q}' \cdot \bar{q}) e^{i\bar{q}' \cdot (\bar{r} - \bar{r}'')} e^{i\bar{q} \cdot (\bar{r}'' - \bar{r}')} \rho(\bar{r}'') \psi(\bar{r}'). \quad (4-20)$$

The exact form of the pion optical model is determined by the treatment of the p-wave scattering amplitude. For the local optical model, the zero-angle approximation is chosen, yielding

$$\mathcal{A} + t \bar{q}' \cdot \bar{q} \rightarrow \mathcal{A} + t q^2. \quad (4-21)$$

In this case, Eq. (4-20) yields the result,

$$V(\bar{r}) \psi(\bar{r}) = (2\pi)^3 A (\mathcal{A} + t q^2) \rho(r) \psi(\bar{r}). \quad (4-22)$$

In the Laplacian model, the amplitude is assumed to depend upon the momentum transfer according to the prescription,

$$\mathcal{A} + t \bar{q}' \cdot \bar{q} \rightarrow \mathcal{A} + t q^2 - \frac{1}{2} t (\bar{q}' - \bar{q}) \cdot (\bar{q}' - \bar{q}). \quad (4-23)$$

Now the potential is given by

$$V(\bar{r}) \psi(\bar{r}) = (2\pi)^3 A \left\{ (\mathcal{A} + t q^2 + \frac{1}{2} t \nabla^2) \rho(r) \right\} \psi(\bar{r}), \quad (4-24)$$

where the Laplacian operates only on the nuclear density function.

Kisslinger,<sup>92</sup> on the other hand, solved Eq. (4-20) to obtain the nonlocal result,

$$V(\bar{r}) \psi(\bar{r}) = (2\pi)^3 A \left\{ \mathcal{A} \rho(r) \psi(\bar{r}) - t \bar{\nabla} \cdot (\rho(r) \bar{\nabla} \psi(\bar{r})) \right\}. \quad (4-25)$$

The local, Laplacian and Kisslinger potentials are summarized in the following equations:

$$-2EV(\vec{r}) = b_0 \rho(r) + b_1 q^2 \rho(r),$$

$$-2EV(\vec{r}) = b_0 \rho(r) + b_1 q^2 \rho(r) + \frac{1}{2} b_1 (\nabla^2 \rho(r))$$

and

$$-2EV(\vec{r}) = b_0 \rho(r) - b_1 \vec{\nabla} \cdot \rho(r) \vec{\nabla}. \quad (4-26)$$

The quantities,  $b_0$  and  $b_1$ , in Eq. (4-26) depend on the transformation chosen in going from the pi-nucleon frame to that of the pi-nucleus. We first make use of the results of Appendix VIII, and summing over isospin, find the average scattering amplitude (for a positive pion) in the pi-nucleon center-of-mass frame to be

$$F_2 = A(\alpha + \beta \vec{q}' \cdot \vec{q}),$$

where

$$\alpha = \frac{1}{q_2^3} \left\{ \frac{(A-Z)(2a_1 + a_3) + 3Za_3}{3A} \right\}$$

and

$$\beta = \frac{1}{q_2^3} \left\{ \frac{(A-Z)(4a_{13} + 2a_{11} + 2a_{33} + a_{31}) + 3Z(2a_{33} + a_{31})}{3A} \right\}. \quad (4-27)$$

The subscript, 2, refers to the two-body (pi-nucleon) frame. In keeping with the models just discussed, spin-flip terms have been neglected in writing Eq. (4-27). The symbol,  $\alpha$ , is defined by

$$a = e^{i\delta} \sin \delta, \quad (4-28)$$

where  $\delta$  is the phase shift. The single subscript,  $2t$ , refers to the s-wave, while  $2t, 2j$  denotes a p-wave term.

In his original calculation, Kisslinger<sup>92</sup> neglected the transformation from the pi-nucleon frame to that of the pi-nucleus. He thus obtained

$$b_0 = 4\pi A \alpha$$

and

$$b_1 = 4\pi A \beta. \quad (4-29)$$

Auerbach,<sup>93</sup> et al, have preferred a transformation diagonal in the partial wave amplitudes. They use

$$b_0 = 4\pi A \frac{q}{q_2} \alpha$$

and

$$b_1 = 4\pi A \frac{q_2}{q} \beta. \quad (4-30)$$

Dedonder,<sup>94</sup> on the other hand, suggests a nondiagonal transformation, and finds

$$b_0 = 4\pi A \frac{g}{g_2} \alpha - (g^2 - g_2^2) b_1$$

where

$$b_1 = 4\pi A \frac{g}{g_2} \beta . \quad (4-31)$$

This form is also preferred by Wilkin.<sup>95</sup>

After Auerbach, et al, we solve a Klein-Gordon equation in which the optical model and Coulomb potentials are introduced in the fourth component. The square result of each is neglected, so that the Klein-Gordon equation becomes

$$\left\{ \nabla^2 + g^2 - 2EV_c(r) - 2EV(r) \right\} \psi(\bar{r}) = 0 . \quad (4-32)$$

This form has the advantage of reducing to the Schrödinger equation at low energy.

As given above, the Kisslinger model potential is nonlocal. However, it has been shown by Krell and Ericson<sup>96</sup> that an equivalent local form may be obtained with the substitution,

$$\psi(\bar{r}) = \frac{\psi'(\bar{r})}{\sqrt{1 - b_1 \rho(r)}} . \quad (4-33)$$

In this case, the Klein-Gordon equation becomes

$$\left[ \nabla^2 + g^2 + (1 - b_1 \rho(r))^{-1} \left\{ -2E V_c(r) + b_0 \rho(r) + g^2 b_1 \rho(r) \right. \right. \\ \left. \left. + \frac{1}{2} b_1 \nabla^2 \rho(r) + \frac{1}{4} (1 - b_1 \rho(r))^{-1} b_1^2 (\bar{\nabla} \rho(r)) \cdot (\bar{\nabla} \rho(r)) \right\} \right] \psi'(r) = 0 .$$

(4-34)

Here the gradient is understood to operate only upon the density function.

Following the usual procedure, one-dimensional equations for the partial waves are obtained. Then the Klein-Gordon equations for the local, Laplacian and Kisslinger forms are given, respectively, by

$$\left[ \frac{d}{dr^2} + g^2 - \frac{\lambda(\lambda+1)}{r^2} - \frac{2EZf}{r} + (b_0 + b_1 g^2) \rho(r) \right] u_\lambda(r) = 0 ,$$

$$\left[ \frac{d^2}{dr^2} + g^2 - \frac{\lambda(\lambda+1)}{r^2} - \frac{2EZf}{r} + (b_0 + b_1 g^2) \rho(r) \right. \\ \left. + \frac{1}{2} b_1 (\nabla^2 \rho(r)) \right] u_\lambda(r) = 0$$

and

$$\left[ \frac{d^2}{dr^2} + g^2 - \frac{\lambda(\lambda+1)}{r^2} - (1 - b_1 \rho(r))^{-1} \left\{ - \frac{2EZf}{r} \right. \right. \\ \left. \left. + (b_0 + b_1 g^2) \rho(r) + \frac{1}{2} b_1 (\nabla^2 \rho(r)) \right. \right. \\ \left. \left. + \frac{1}{4} b_1^2 (1 - b_1 \rho(r))^{-1} (\bar{\nabla} \rho(r)) \cdot (\bar{\nabla} \rho(r)) \right\} \right] u'_\lambda(r) = 0 .$$

(4-35)

In these equations the Coulomb potential has been written as a point interaction.

To solve these equations, a general optical model code was written which incorporates the foregoing forms and transformations. The integration technique has been described in Appendix III. In the calculation, the scattering matrix is determined by matching trial wave functions to Coulomb functions at  $r$  and  $r-\epsilon$ . The elastic cross section is then given by<sup>97</sup>

$$\sigma_e = \frac{\pi}{q^2} \sum_{\lambda=0} (2\lambda+1) |1 - S_\lambda|^2, \quad (4-36)$$

while the total cross section is obtained from

$$\sigma_t = \frac{2\pi}{q^2} \sum_{\lambda=0} (2\lambda+1) (1 - \text{Re } (S_\lambda)). \quad (4-37)$$

The angular distribution is given by

$$\frac{d\sigma_e}{d\Omega} = \frac{1}{4q^2} \left| \sum_{\lambda=0} (2\lambda+1) (S_\lambda - 1) P_\lambda(\cos\theta) \right|^2. \quad (4-38)$$

The nuclear density function for this study is obtained from the deuteron S-state wave function, taking into account the pi-nucleon interaction distance through a Gaussian form. Thus the optical model density is given by



$$\rho(r) \sim \int d\vec{r}' e^{-\frac{|\vec{r}-\vec{r}'|^2}{2\sigma^2}} \frac{u^2(2r')}{r'^2}, \quad (4-39)$$

where  $\sigma^2$  is a measure of the pi-nucleon interaction distance. The argument of the S-state wave function reflects the nucleon-nucleon separation. Since  $u(2r')$  is a spherically symmetric function, the integral above is easily shown to be

$$\rho(r) \sim e^{-\frac{r^2}{2\sigma^2}} \int dr' e^{-\frac{r'^2}{2\sigma^2}} \left( e^{\frac{rr'}{2\sigma^2}} - e^{-\frac{rr'}{2\sigma^2}} \right) \frac{u^2(2r')}{rr'}. \quad (4-40)$$

This integral is obtained numerically making use of polynomial expansions for the exponential terms in  $r'$ . The result is

$$\rho(r) \sim e^{-\frac{r^2}{2\sigma^2}} \sum_{i=1}^N c_i r^{2i-2},$$

where

$$c_i = \frac{1}{(2i-1)!} \int dr' e^{-\frac{r'^2}{2\sigma^2}} \left( \frac{r'}{\sigma^2} \right)^{2i} \frac{u^2(2r')}{r'^2}. \quad (4-41)$$

Finally, the density function is normalized according to Eq. (4-19).

Plots of this optical model density for several values of  $\sigma^2$  are shown

in Fig. 16. Since the density function is spherically symmetric, use is made of the relations,

$$\nabla^2 \rho(r) = \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \rho(r)$$

and

$$(\bar{\nabla} \rho(r)) \cdot (\bar{\nabla} \rho(r)) = \left( \frac{d}{dr} \rho(r) \right)^2, \quad (4-42)$$

in numerically evaluating these terms in the potential.

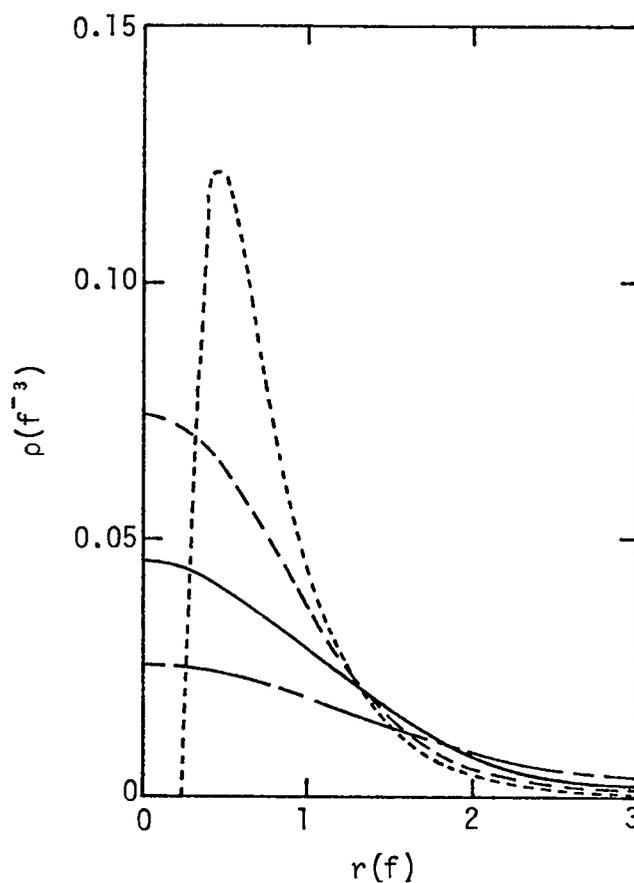


Fig. 16. Deuteron density functions for the pion optical model.

-----	$(u(2r)/r)^2$	- - - - -	$\sigma^2 = 0.2f^2$
—————	$\sigma^2 = 0.5f^2$	- - - - -	$\sigma^2 = 1.0f^2$

Figure 17 shows results of total cross section calculations using the local, Laplacian and Kisslinger forms compared with experimental data.<sup>69,98-103</sup> The Auerbach kinematical transformation and the interaction distance parameter,  $\sigma^2 = 0.5f^2$ , are used. Calculated angular distributions and experimental data<sup>69,102,103</sup> at three energies are shown in Fig. 18. The Kisslinger and Laplacian models are seen to give very similar results. The pronounced minima near ninety degrees may be associated with omission of spin-flip terms in the scattering amplitude. Since this term vanishes at forward and backward angles, results at these angles are felt to be a better measure of the adequacy of this calculation. The local optical model, on the other hand, exhibits its usual behavior in that its minimum lies at a larger angle.

We turn now to a second-order optical model formalism for pi-deuteron scattering. Use of the approximate result of Eq. (4-16) can be shown equivalent to elimination of the sum restrictions in Eq. (4-9). This approximation seems plausible for large nuclei. For the deuteron, however, Eq. (4-16) would result from the multiple scattering series,

$$T = t_p + t_n + t_p \bar{a}' t_p + t_p \bar{a}' t_n + t_n \bar{a}' t_p + t_n \bar{a}' t_n + \dots, \quad (4-43)$$

which represents significant error in second order. We write the wave function obtained using the above series symbolically as  $\psi(t_p + t_n)$ . Results presented up to this point were obtained using this calculation. On the other hand, the desired result, to second order, is given by

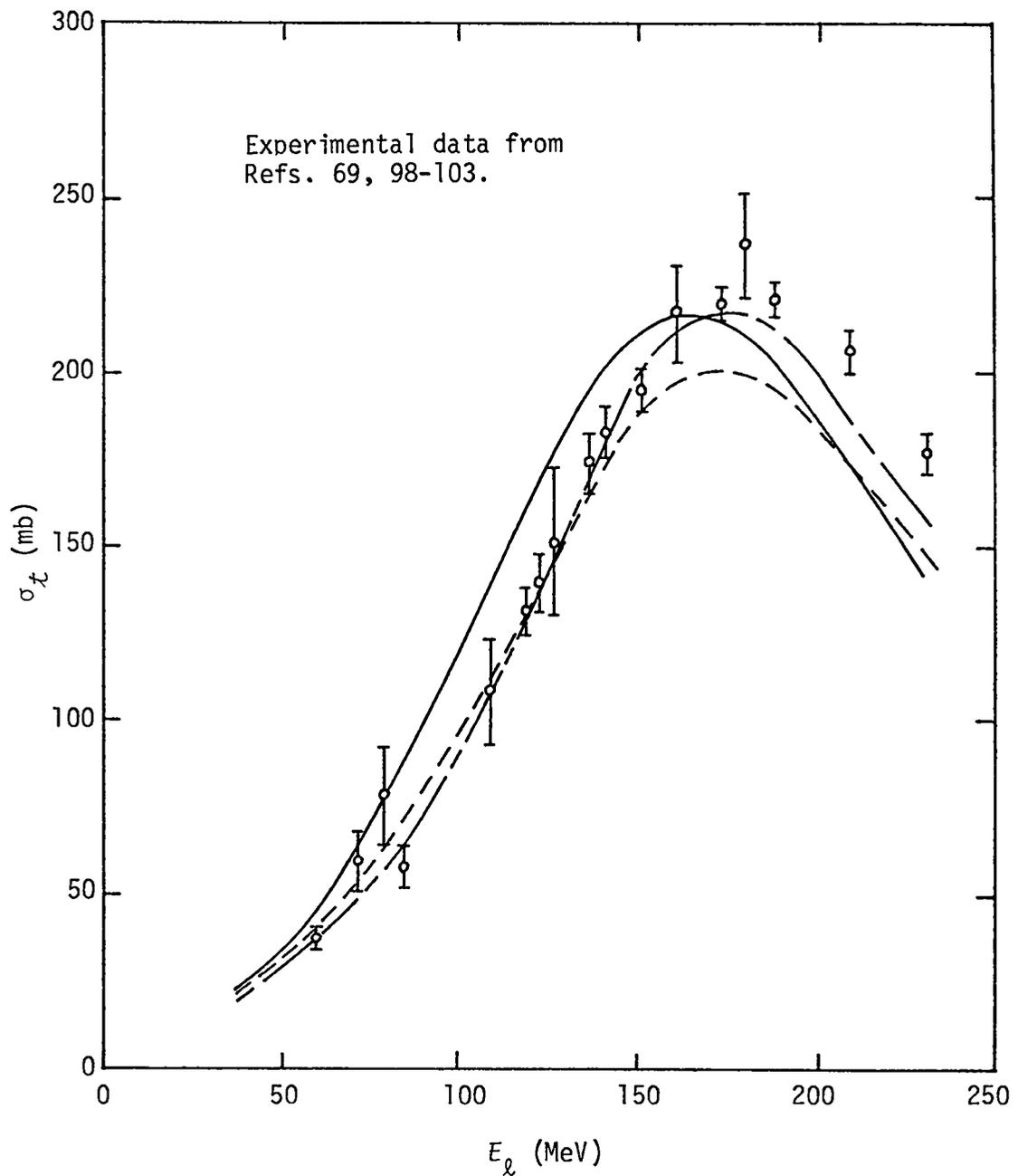


Fig. 17. Total scattering cross sections for general formalism with Auerbach transform and  $\sigma^2 = 0.5 f^2$ .

- local
- . - . - Laplacian
- Kisslinger

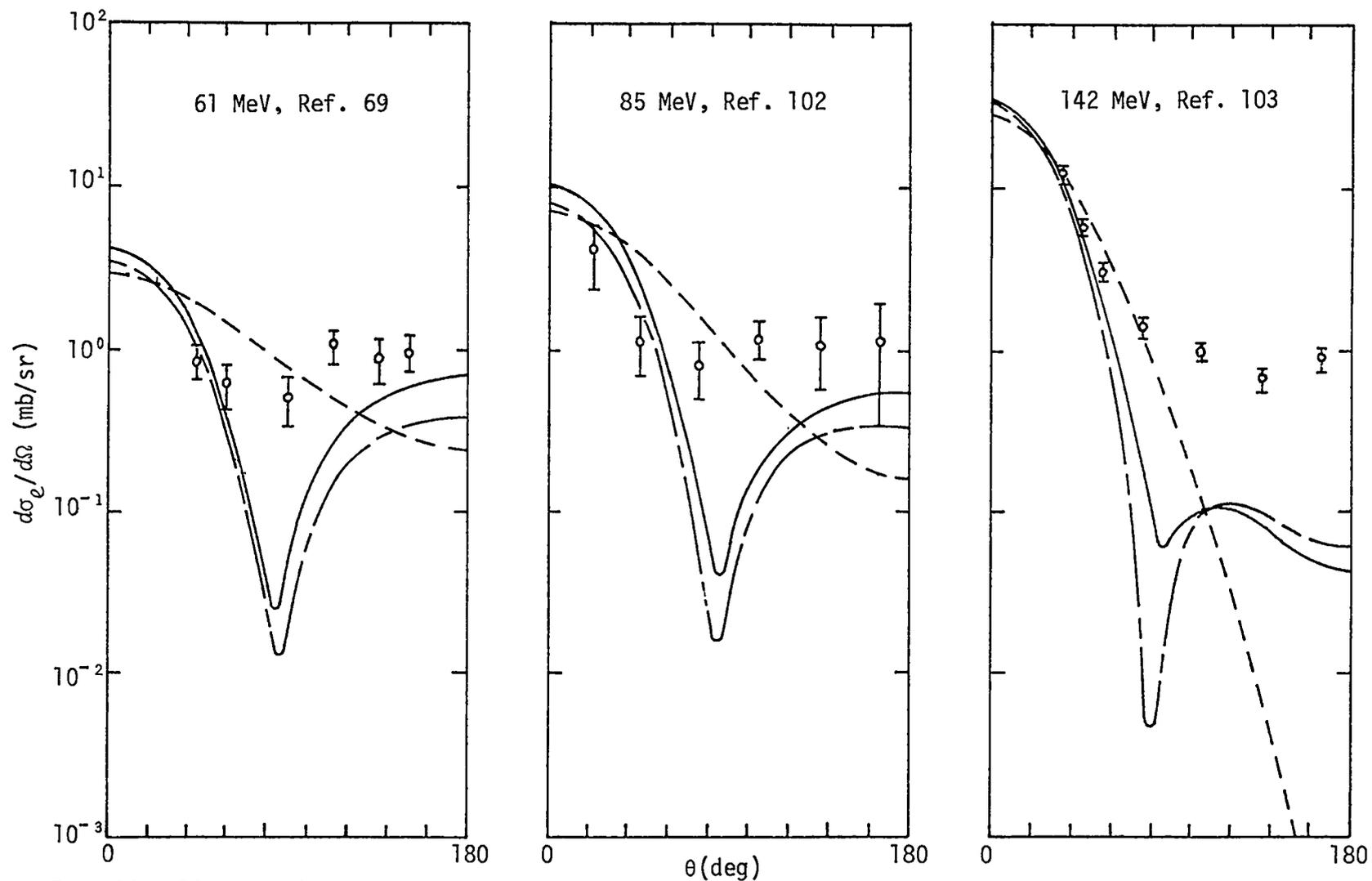


Fig. 18. Differential scattering cross sections at 61, 85 and 142 MeV for general formalism with Auerbach transform and  $\sigma^2 = 0.5f^2$ .

----- local      - · - · - Laplacian      ——— Kisslinger

$$T = t_p + t_n + t_p \bar{a}' t_n + t_n \bar{a}' t_p + \dots \quad (4-44)$$

This series can be summed by assuming the following linear combination of scattering operators:

$$1 + T = 1 + (1 + T_1) - (1 + T_2) + (1 + T_3) - (1 + T_4) \quad (4-45)$$

The potentials corresponding to these operators are in turn given by a linear combination of neutron and proton scattering amplitudes. Thus the wave function,

$$\psi = (1 + \bar{a}' T) \phi, \quad (4-46)$$

can be written as

$$\psi = \phi + \psi_1\left(\frac{t_p + t_n}{2}\right) - \psi_2\left(\frac{-t_p + t_n}{2}\right) + \psi_3\left(\frac{t_n + t_p}{2}\right) - \psi_4\left(\frac{-t_n + t_p}{2}\right) \quad (4-47)$$

The prescription now to obtain the wave function is to solve the Klein-Gordon equation for each of the potentials in Eq. (4-47), then to take the indicated sum of the resulting functions. Use is made of the fact that  $V_3 = V_1$ ; however, spherical Bessel functions or Coulomb functions

are most easily generated with the Klein-Gordon equation and zero potential. Thus, in practice, four integrations are required. The S-matrix element is easily seen to be

$$S_{\lambda} = 1 + 2S_{1\lambda} - S_{2\lambda} - S_{4\lambda} . \quad (4-48)$$

Calculations were performed using this second-order formalism with the previously described forms and transformations. Figure 19 illustrates total cross sections obtained using the kinematical transformation of Dedonder and an interaction parameter of  $\sigma^2 = 0.5f^2$ . This result is seen to offer improvement over that of the general formalism at higher energies. An additional interesting effect is the apparent model independence. Angular distributions obtained from these calculations are shown in Fig. 20. Again an improvement is noted, here in the back-angle scattering. Roughly half of this improvement can be attributed to the second-order formalism; the remainder is due to use of the Dedonder transform.

Figure 21 presents the results of an investigation of pi-nucleon size using different values of the interaction distance parameter,  $\sigma^2$ . Reasonable agreement at back-angle is obtained using  $\sigma^2 = 0.3f^2$ , a result comparable with the off-shell parameter value,  $\alpha = 300$  MeV/c, obtained in Chapter 2. This value of  $\sigma^2$  is therefore used in the absorption calculations to be presented here. The scattering results shown in Fig. 21 are obtained using the Kisslinger optical model and the Dedonder transformation. It is perhaps worth noting that if one had confidence in the validity of the calculation, this technique could be inverted and

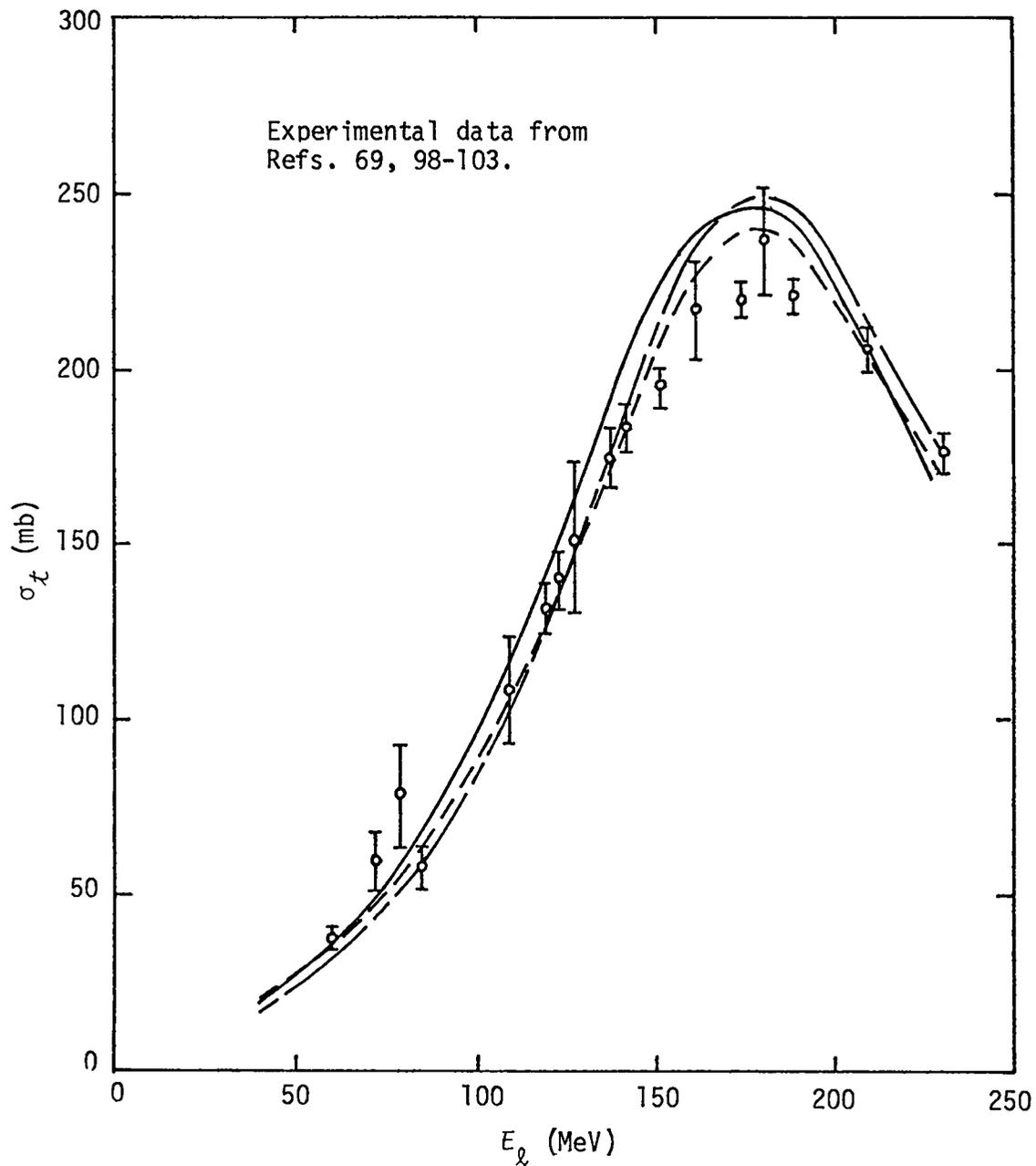


Fig. 19. Total scattering cross sections for second-order formalism with Dedonder transform and  $\sigma^2 = 0.5 f^2$ .

- - - - - local  
 - · - · - Laplacian  
 ——— Kisslinger



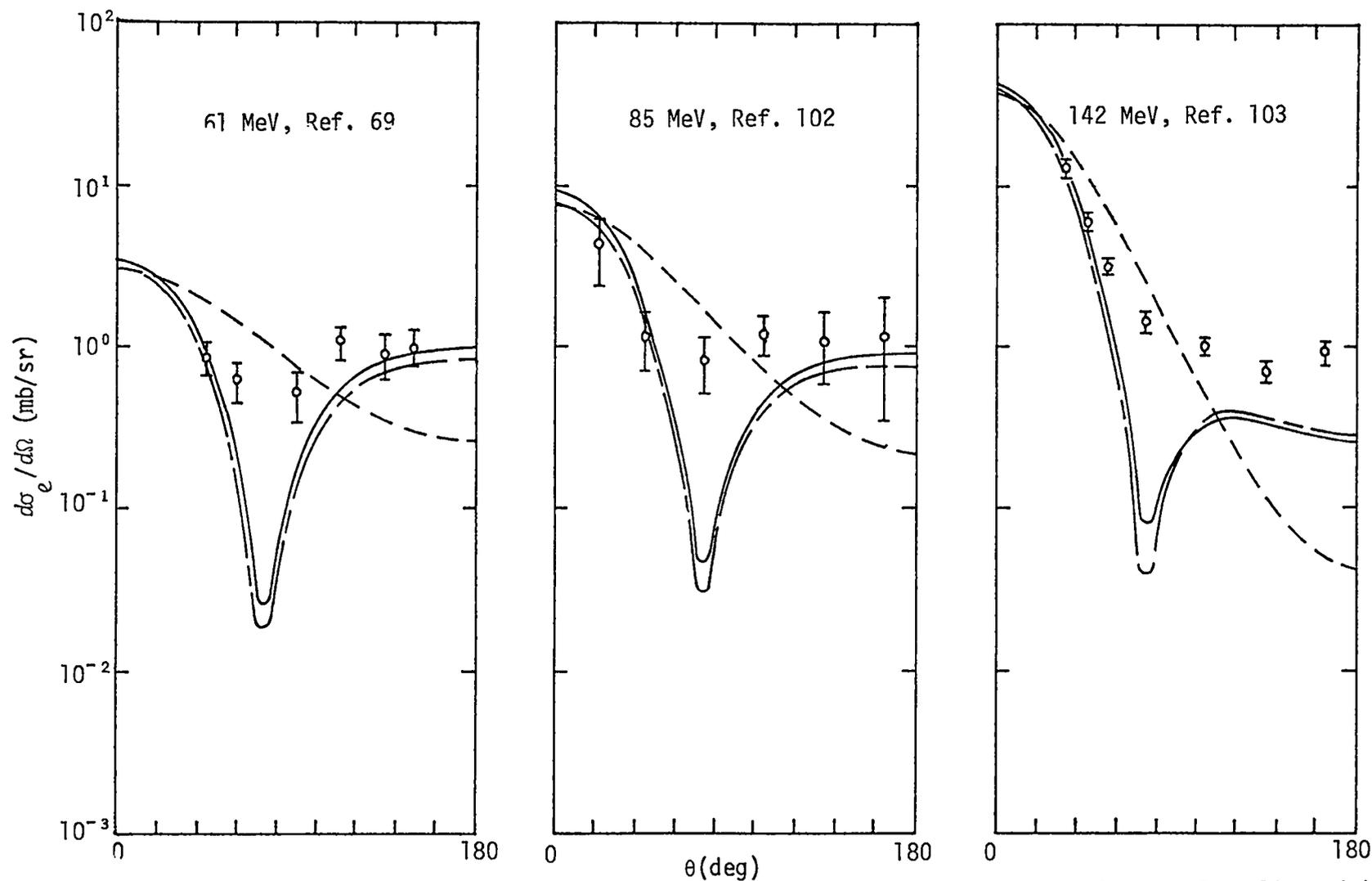


Fig. 20. Differential scattering cross sections at 61, 85 and 142 MeV for second-order formalism with Dedonder transform and  $\sigma^2 = 0.5 f^2$ .

----- local      - · - · - Laplacian      ——— Kisslinger

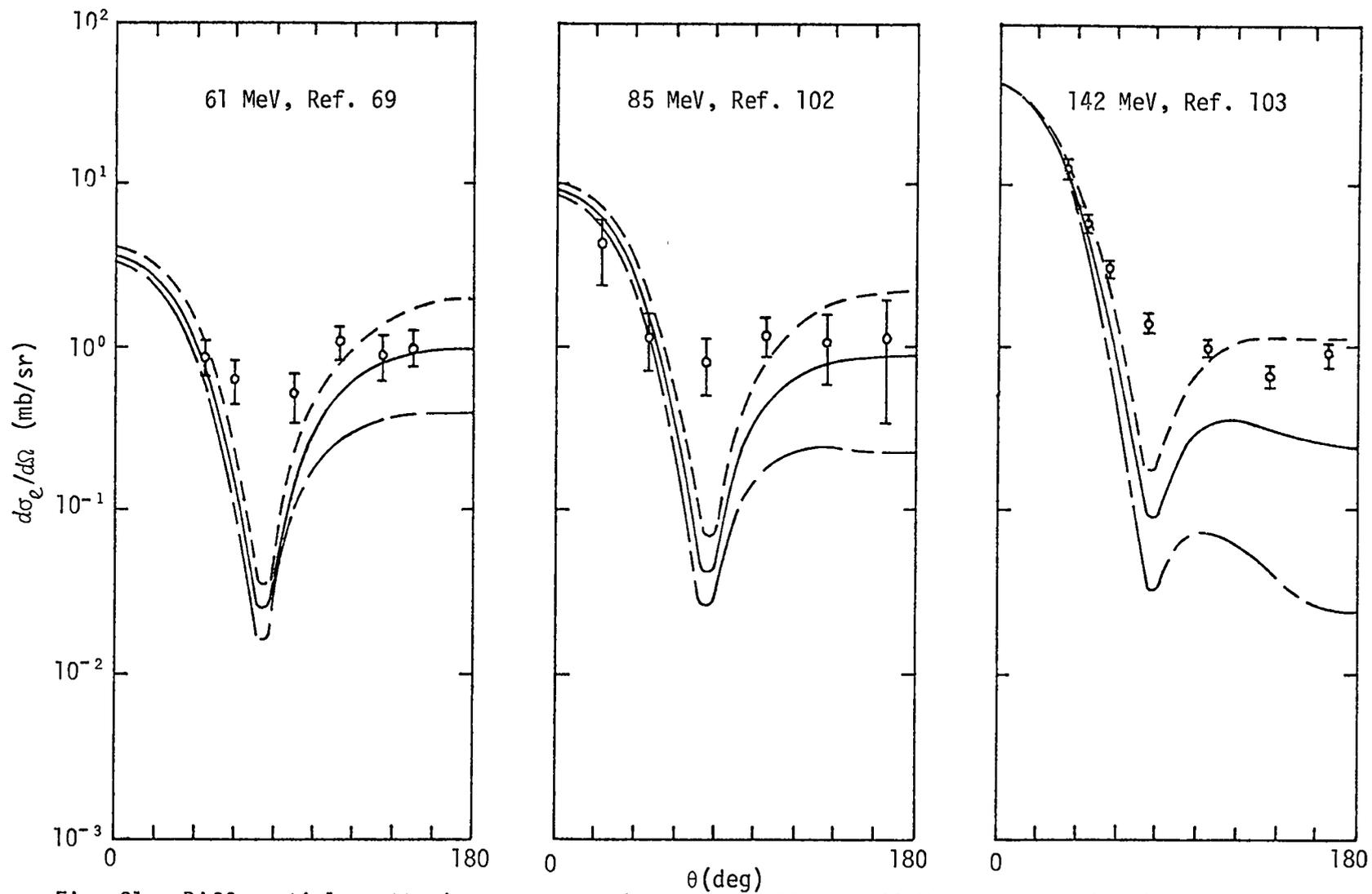


Fig. 21. Differential scattering cross sections at 61, 85 and 142 MeV for second-order formalism with Kisslinger model and Dedonder transform.

-----  $\sigma^2 = 0.2 f^2$

—————  $\sigma^2 = 0.5 f^2$

- · - · -  $\sigma^2 = 1.0 f^2$

the pi-nucleon interaction distance thereby determined. However, the intent here is simply to establish the credibility of these second-order pion wave functions for use in the absorption problem, to which we now turn.

In the two preceding chapters, pi-nucleon scattering has been demonstrated to dominate the reaction,  $\pi^+ + d \rightarrow p + p$ , throughout the energy range. Now the essential idea will be to incorporate this scattering through an optical model. Formally, this corresponds to modification of the pion field in the Hamiltonian, as opposed to the usual procedure of adding terms to the Hamiltonian. Specifically, we retain in the Hamiltonian presented in Chapter 2 only those terms linear in the pion field. Then the nuclear transition operator is given by the direct absorption operator,  $T^0$ , of Eq. (2-7). This result is repeated here for convenience:

$$\begin{aligned}
 T^0 = & - (4\pi)^{1/2} f \bar{\mu}' \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{R}} \\
 & \left\{ \left( e^{i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_1^+ \bar{\sigma}_1 + e^{-i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_2^+ \bar{\sigma}_2 \right) \cdot \left( 1 + \frac{g_0}{2M} \right) \bar{q} \right. \\
 & \left. - \left( e^{i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_1^+ \bar{\sigma}_1 - e^{-i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_2^+ \bar{\sigma}_2 \right) \cdot \left( \frac{g_0}{M} \right) \bar{k}' \right\}. \quad (4-49)
 \end{aligned}$$

It is now apparent that  $\bar{q}$  (as well as  $\bar{k}'$ ) must be cast as an operator. The partial wave expansion of the external pion field remains valid; thus the essential replacement in the integrals in Eq. (2-23) is that of

$$j_\lambda\left(\frac{qr}{2}\right) \longrightarrow \frac{u_\lambda(r/2)}{r/2} = \psi_\lambda\left(\frac{r}{2}\right). \quad (4-50)$$

The four required integrals are then given by

$$\begin{aligned}
 I_1 &= -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} - \frac{\lambda}{r}\right) \psi_\lambda, \\
 I_2 &= -2\left(1 + \frac{g_0}{2M}\right) \int dr r^2 \frac{u}{r} g'^* \left(\frac{d}{dr} + \frac{\lambda+1}{r}\right) \psi_\lambda, \\
 I_3 &= -\frac{g_0}{M} \int dr r^2 \frac{u}{r} \psi_\lambda \left(\frac{d}{dr} - \frac{L'}{r}\right) g'^*
 \end{aligned}$$

and

$$I_4 = -\frac{g_0}{M} \int dr r^2 \frac{u}{r} \psi_\lambda \left(\frac{d}{dr} + \frac{L'+1}{r}\right) g'^*, \quad (4-51)$$

and the matrix element coefficient by

$$C(S'M'M\lambda) = i^{\lambda+1} (4\pi)^{3/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} \sum_{J'L'L} \sum_{\lambda=1}^4 R_\lambda I_\lambda. \quad (4-52)$$

The angular integrals,  $R_\lambda$ , are unchanged from the result given in Appendix VI. From Eq. (4-49) onward, the symbol,  $r$ , refers to the nucleon-nucleon separation.

For the absorption calculation, we write the pion wave function in terms of three components:

$$\psi = \phi + \psi_d + \frac{1}{\sqrt{2}} \psi_e. \quad (4-53)$$

The plane wave component is easily identified with the direct absorption graph shown on the left in Fig. 22. In the remaining terms and figures we differentiate between the processes of direct- and charge-exchange scattering.

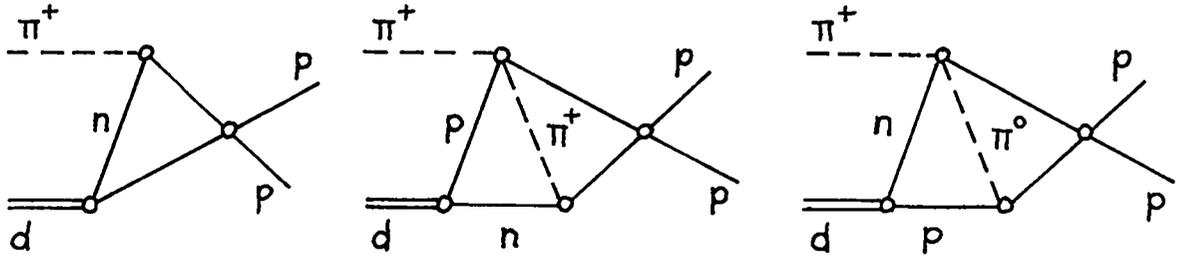


Fig. 22. Absorption processes in the optical model formalism.

Using the second-order method previously given, the direct-exchange scattered wave function is found to be

$$\psi_d = \psi\left(\frac{t_p + t_n}{2}\right) - \psi\left(\frac{-t_p + t_n}{2}\right). \quad (4-54)$$

The center graph in Fig. 22 is taken to be symbolic of this multiple-scattering process. For the remaining scattered component of the wave function,  $\psi_e$ , a charge-exchange amplitude,  $t_e$ , is required. This has been evaluated in Appendix VIII. The scattered wave function is given correctly to second order by

$$\psi_e = \psi\left(\frac{t_e + t_p}{2}\right) - \psi\left(\frac{-t_e + t_p}{2}\right). \quad (4-55)$$

In Eq. (4-53), this component has been reduced by a root-two factor to reflect a reduced amplitude at the absorption vertex for this process.

By incorporating the above results in Eq. (4-53), we obtain

$$\psi = \phi + \psi\left(\frac{t_p+t_n}{2}\right) - \psi\left(\frac{-t_p+t_n}{2}\right) + \frac{1}{\sqrt{2}}\left\{\psi\left(\frac{t_e+t_p}{2}\right) - \psi\left(\frac{-t_e+t_p}{2}\right)\right\}.$$

(4-56)

It should be noted that, while scattering has been included to second order, the backward-scattering graph shown in Fig. 3 has been completely omitted. In the pure Hamiltonian formalism, of course, the meson is virtual, while in the present optical model formalism the interpretation must reflect a physical pion.

Using the pion wave function of Eq. (4-56), calculations were performed as outlined above for the absorption reaction. Figure 23 presents results obtained for the local, Laplacian and Kisslinger forms of the optical model. In all three forms, the Dedonder transformation and a value,  $\sigma^2 = 0.3f^2$ , are used. Four partial waves in the pion wave function are included. Pi-nucleon amplitudes are evaluated according to previously given relations using the phase shifts of McKinley.

We find the interesting result that the optical model formalism does succeed in predicting an absorption resonance. (The Laplacian resonance occurs at very high energy.) For contrast, the plane wave result previously shown in Fig. 4 is reproduced in Fig. 23. The effect introduced through the pion scattering is now made more apparent. The local model is observed to give the best position for the resonance, although its magnitude is not sufficient. Use of the Auerbach transformation enhances this result. Also notable is the difference

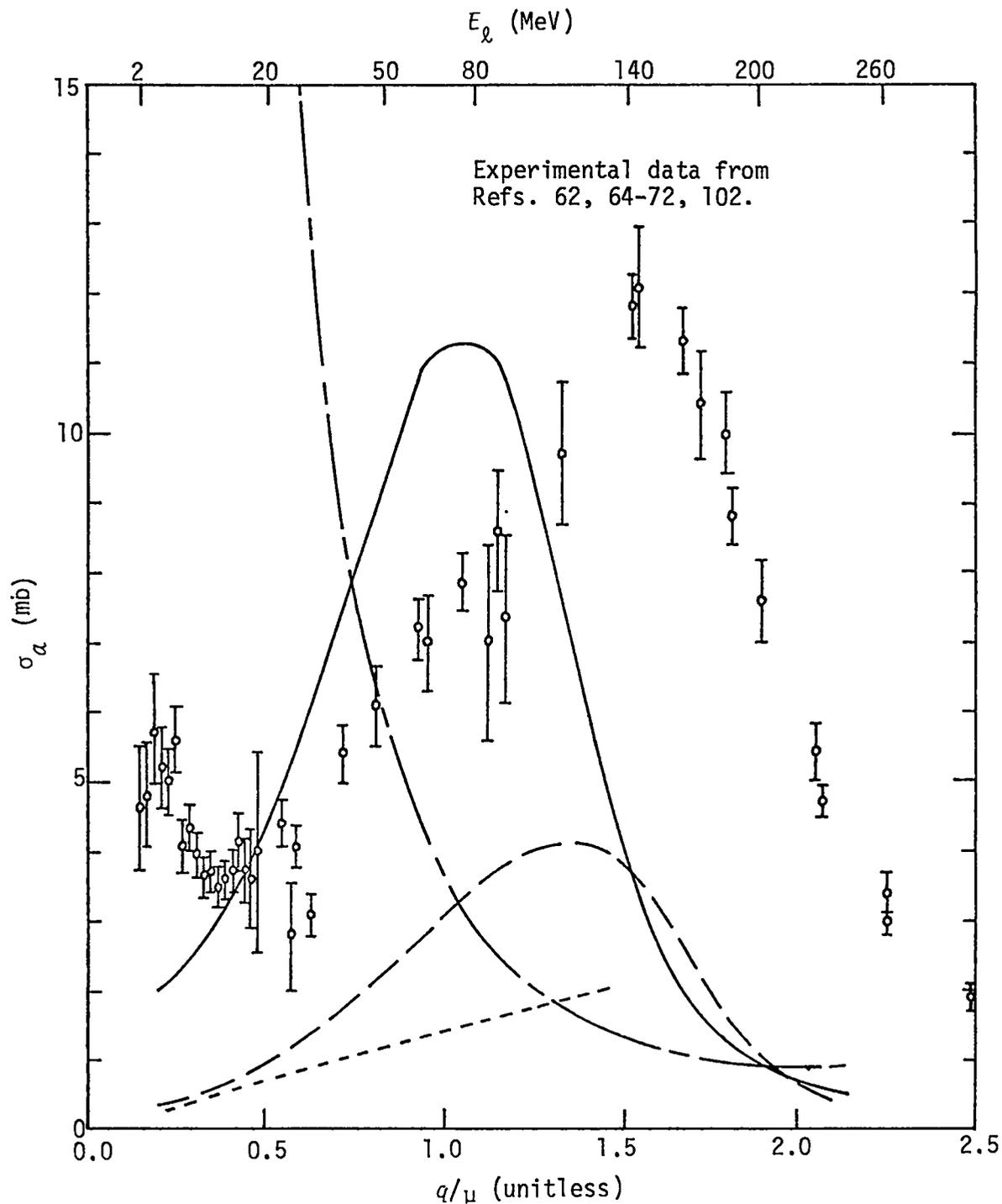


Fig. 23. Absorption cross sections for the pion optical model formalism.

-----	plane wave	- - - - -	local
- . - . -	Laplacian	—————	Kisslinger

obtained between the Kisslinger and Laplacian models. This is especially surprising in light of the similarity of the scattering results. Examination of the Kisslinger wave functions, however, indicates an oscillatory behavior at short range which contributes significantly to the integrals. The local model wave functions seem to bear the closest resemblance to the spherical Bessel functions. The divergence in the case of the Laplacian model, however, is associated with the lowest partial wave. Angular distributions obtained from the local and Kisslinger models appear qualitatively reasonable, having large  $\cos^2\theta$  components in the vicinity of the resonance.



## CHAPTER 5

### CONCLUSION

The nonradiative absorption of positive pi-mesons by deuterons has been studied within the framework of Lagrangian formalism and the impulse approximation. Pion absorption on the deuteron is dominated by pi-nucleon scattering, and this investigation has focused mainly upon different approaches to scattering processes in the absorption calculations.

Our first approach to the absorption reaction involved extension of the Hamiltonian formalism of Koltun and Reitan to include energetic pions. The main modifications to this formalism are the addition of graphs for p-wave pi-nucleon scattering and the inclusion of all partial waves of the incident pion. At low energy, for which only s-wave scattering and direct absorption terms are significant, good agreement is obtained with the original bound-state results of Koltun and Reitan. The well-known cancellation of deuteron S- and D-state direct terms emphasizes the importance of pi-nucleon scattering even at low energy. At very low energy, the cross section is dominated by s-wave scattering graphs, and reasonable agreement with experimental data is obtained. However, the s-wave scattering result falls off with increasing energy, while experimental values increase after 20 MeV up to the resonance. It is therefore necessary to include p-wave scattering graphs even at low incident pion energy.

Several forms of the p-wave scattering Hamiltonian were examined, including the low-energy result given by Klein. Use of this p-wave

Hamiltonian leads to calculated cross sections greatly in excess of the experimental data, even at low incident pion energy. The intermediate state includes integrals over all momenta of the virtual meson; the Klein low-energy form tends to overemphasize high-momentum components in these integrals. Attempts to reduce high-momentum contributions by means of momentum cutoffs in the intermediate-state integrals are unsuccessful because of sensitivity to the cutoff parameters.

As an alternate means of reducing high-momentum contributions, an ad hoc alteration of the Klein p-wave Hamiltonian was proposed. This alteration consists of a reduction in power for the momentum operator of the virtual meson. The resulting Hamiltonian is asymmetric; however, it is seen to be equivalent below the resonance to the on-shell scattering amplitude. (This calculation is therefore similar to that obtained from the S-matrix formalism of Lazard, Ballot and Becker.) Calculated total cross sections up to 100 MeV show good agreement with experimental data.

In all approaches to the absorption reaction, a Galilean-invariant absorption Hamiltonian is assumed. However, using the on-shell calculation, we find that Galilean invariance cannot be confirmed using the pi-deuteron absorption reaction. Again, this directly reflects the accidental cancellation of S- and D-state direct absorption terms.

For problems involving high-momentum off-shell processes, neither of the scattering forms discussed above has a sound theoretical basis. First, the low-energy result of Klein does not vanish in the high-momentum limit. The on-shell Hamiltonian also suffers from this defect and, in addition, does not reduce to Klein's result at low energy. On the other

hand, the separable form of the p-wave Hamiltonian given in Eq. (2-5) has these desired properties, and is thus felt to be more suitable for investigation of the absorption reaction. Calculated cross sections obtained with this Hamiltonian show great sensitivity to the off-shell parameter,  $\alpha$ . Good agreement with experimental data is obtained with a value,  $\alpha = 300 \text{ MeV}/c$ , which lies midway in the range of values suggested by other investigators. The sensitivity of calculated results to this parameter is due in part to the form of the p-wave absorption Hamiltonian, which brings an extra power in intermediate pion momentum into the matrix element. For this reason, pi-deuteron absorption offers an excellent means of studying the off-shell behavior of pi-nucleon scattering.

It is also interesting that some sensitivity to the nuclear wave function is retained in this calculation. Differences between potentials show up mainly in the angular distributions, and seem significant enough to allow one to select a preferred model given better experimental data. The calculations performed indicate a sensitivity to the fraction of D-state in the deuteron. This possibility might be confirmed with calculations using other potentials. Given the magnitude and approximate cancellation of S- and D-state direct terms, it is somewhat surprising that calculated results are not more strongly dependent upon the fraction of D-state. This effect may be explicable in terms of the deuteron quadrupole moment, and it is worth noting that this quantity is well represented by all of the deuteron wave functions used here, including the pole models. All are similarly seen to reasonably reproduce the experimental absorption data.

In our second approach to the absorption problem, the S-matrix formalism of Lazard, Ballot and Becker was modified for use with physical wave functions. A dominant feature of this formalism is the direct inclusion of on-shell scattering amplitudes in the matrix element. This calculation therefore bears much similarity to the on-shell Hamiltonian approach. The similarity may be observed term-for-term by contrasting coefficients of the integrals of Eq. (2-23) with those of Eq. (3-24). Below the resonance, differences arise from the kinematical factor,  $\tilde{W}/\sqrt{k_0^2 P_0}$ , which appears in all scattering terms, and from the evaluation of isospin contributions to s-wave scattering. These factors are more important than the more precise calculation of propagator terms, and an improvement in low-energy results over those for the on-shell Hamiltonian is noted.

Total cross sections calculated using LBB approximation I and physical wave functions give good agreement with experimental data throughout the energy range. The agreement appears worse when all operator terms and pion partial waves are included in the calculation. Further, these results show positive coefficients for higher order terms in the angular distribution. This problem may be related to nonunitarity in the  $^3P_1$  and  $^1D_2$  proton-proton states at higher energy. (Also noteworthy is the importance of nonunitarity in the original LBB calculations in reducing calculated cross sections to experimental values. Wave functions calculated from real, phenomenological potentials are, of course, unitary; yet an equivalent reduction in cross section using these wave functions is obtained.) It should be pointed out that we are using these nucleon-nucleon potentials at energies in considerable excess of those for which they were constructed. S-matrix

calculations in which the two states in question are simply renormalized show considerable improvement in the angular results. However, in keeping with the spirit of these calculations, a correct approach would involve the inclusion of imaginary terms in the nucleon-nucleon potential.

In addition to the nuclear absorption difficulty, there are other problems at high energy associated with both the Hamiltonian and S-matrix approaches. These include pi-nucleon scattering contributions from terms higher than p-wave, contributions from multiple scattering graphs, and questions related to the credibility of basic operators, such as the off-shell behavior of the direct absorption Hamiltonian. These complicating factors should promote renewed interest in the low-energy region where the situation is somewhat more clear. In this regard, we again point out the proposed experiment at LAMPF, and note that the angular distributions to be obtained may allow selection of a preferred nuclear model (Fig. 7). It is interesting that the S-matrix formalism provides a very similar result (Fig. 13). This particular similarity occurs because of the nearly equivalent treatment of s-wave scattering, which dominates the reaction at low energy. However, there is no reason in principle why the s-wave Hamiltonian of Eq. (2-3) should not have off-shell dependence similar to that of the p-wave term. Such dependence would be expected to show less sensitivity to an off-shell parameter simply because of the lower powers of momentum found in the s-wave intermediate integrals. A more difficult problem at low energy involves contributions from p-wave scattering, which in the present formulations vanish in the zero incident-momentum limit. Such p-wave contributions arise from Fermi motion of the nucleons, and are nonvanishing

even for the bound pion case. This effect might amount to as much as 25 percent at low energy.

The general optical model formalism has been applied with reasonable success to the problem of pi-deuteron scattering, and results exhibit many characteristics derived for scattering from larger nuclei. The second-order formalism developed for the deuteron is also successful in reproducing the scattering data, and further seems to give greater model independence. Minima in the differential cross sections have been associated with omission of the spin-flip terms from the scattering amplitudes. Satisfactory back-angle scattering results are achieved with a reasonable value of the parameter characterizing pi-nucleon range. It has been noted that, in principle, the scattering result could be used to determine this parameter.

There is little evidence that the optical model formalism offers a fundamental calculation for the pi-deuteron absorption problem. It remains interesting, however, for several reasons. First, a moderately successful representation of the absorption data (i.e., a resonance) is achieved using a physical meson in the intermediate state. We contrast this with the Hamiltonian and S-matrix formalisms in which the scattered meson is highly virtual. Also, second-order scattering graphs are included in this absorption calculation. Such a graph consists of a coherent scattering on the first nucleon, followed by an incoherent scattering on the second (leading to breakup) and absorption on the first nucleon. In addition, the optical model formalism offers a simpler absorption calculation in that all intermediate momentum integrals and most of the tedious angular and one-dimensional radial integrals are eliminated. Finally, because of the sensitivity exhibited to

optical model parameters, the absorption calculation may provoke some new insight into the pion optical model. In this regard we note the dissimilarity in results obtained with the Laplacian and Kisslinger models, as well as sensitivity to parameters such as the kinematical transformation and the pi-nucleon interaction distance.

In conclusion there are several points to be emphasized. The first is that calculated results, and in particular angular distributions, show sufficient sensitivity to nucleon-nucleon wave functions to allow selection of a preferred potential model. Such sensitivity may extend to a determination of the D-state fraction in the deuteron.

We have also demonstrated the extraordinary sensitivity of calculated cross sections to the off-shell pi-nucleon scattering behavior. We suggest that the pi-deuteron absorption reaction offers an important means of investigating these off-shell effects.

Finally, there remains the question of whether pion absorption reactions can be useful in the investigation of nuclear structure. In this regard it seems important that a reasonable representation of the pi-deuteron data can be obtained using realistic wave functions and credible forms for the interaction Hamiltonian. This would seem to offer hope for the use of similar techniques in the investigation of the more complex nuclei. Since the pi-deuteron reaction is dominated by scattering processes, it seems likely that such processes also play an important role in the general nuclear case. For such nuclear structure investigations, we recommend use of the p-wave Hamiltonian of Eq. (2-5).

## APPENDIX I

### THE NUCLEAR TRANSITION OPERATOR

The nuclear transition operator,  $T$ , is evaluated using the Hamiltonian interaction densities discussed in Chapter 2. We have established the convention of using subscripts for the nucleon numbers and superscripts for tensor components of vectors. However, the isospin and pion-field components differ in sign from the usual tensor definition; this exception is necessary since Mandl<sup>104</sup> uses the form,

$$\phi^\pm = +\frac{1}{\sqrt{2}}(\phi^1 \pm i\phi^2). \quad (\text{I-1})$$

Mandl's expressions for pion field and conjugate are given by

$$\begin{aligned} \phi^+ &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{1}{2q_0}} \left\{ a_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} + b_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\}, \\ \phi^- &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{1}{2q_0}} \left\{ b_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} + a_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\}, \\ \phi^3 &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{1}{2q_0}} \left\{ c_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} + c_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\}, \\ \pi^+ &= \frac{i}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{q_0}{2}} \left\{ a_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} - b_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\}, \\ \pi^- &= \frac{i}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{q_0}{2}} \left\{ b_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\} \end{aligned}$$

and

$$\pi^3 = \frac{i}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{q_0}{2}} \left\{ c_{\mathbf{q}}^+ e^{-i\mathbf{q}\cdot\mathbf{x}} - c_{\mathbf{q}}^- e^{+i\mathbf{q}\cdot\mathbf{x}} \right\}. \quad (\text{I-2})$$



In these expressions,  $a$ ,  $b$  and  $c$  are creation (+) and destruction (-) operators for positive, negative and neutral pi-mesons, respectively. The superscript stating the field component thus also reflects the effect of the field upon total meson charge. (The superscript, three, might also be written as zero.)

We begin by considering processes involving direct absorption. The nuclear transition operator is given in Eq. (2-1) as

$$T^0 = \langle 0 | \sum_{i=1}^2 H_i^0 | \bar{\psi}_i, c=+1 \rangle. \quad (\text{I-3})$$

We first consider the process of direct absorption on nucleon (1), as represented by the graph in Fig. 24. In this appendix, graphs will be labeled to indicate the four-momenta of the particles.

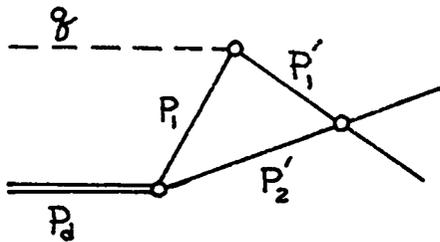


Fig. 24. Graph for direct absorption on nucleon (1).

Subscripts on the transition operator symbol will be used to denote the particular process under consideration. In this case, for example, we have

$$T_1^0 = \langle 0 | H_1^0 | \bar{\psi}_1, c=+1 \rangle. \quad (\text{I-4})$$

We may also replace the pion momentum operator in the absorption Hamiltonian in Eq. (2-2) with a gradient operator on the pion field, according to

$$\bar{q}_{\pi} = -i \bar{\nabla}_{\pi} . \quad (I-5)$$

Making use of the above relations and Eq. (2-2), we can write

$$\begin{aligned} T_1^0 = & \langle 0 | (4\pi)^{1/2} f_{\mu}^{-1} i \\ & \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_{\pi} \tau \cdot \phi + (2M)^{-1} (\bar{P}_1 \tau \cdot \pi + \tau \cdot \pi P_1) \right\} | \bar{q}, c=+1 \rangle . \end{aligned} \quad (I-6)$$

The isospin-field dot products can be expanded into vector or tensor-like components, and are of the form,

$$\tau \cdot \phi = \tau^1 \phi^1 + \tau^2 \phi^2 + \tau^3 \phi^3,$$

or

$$\tau \cdot \phi = \tau^+ \phi^- + \tau^- \phi^+ + \tau^3 \phi^3. \quad (I-7)$$

The field in  $T_1^0$  is seen to be linear. Thus all terms not containing  $\tau^+ \phi^-$  or  $\tau^- \phi^+$  vanish, and in these, only terms involving positive meson destruction can contribute. The operator  $T_1^0$  is therefore given by

$$T_1^0 = (4\pi)^{1/2} f \bar{\mu}^{-1} i$$

$$\langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \pi_1^+ \phi_1^- + (2M)^{-1} (\bar{p}_1' \pi_1^+ \pi_1^- + \pi_1^+ \pi_1^- \bar{p}_1) \right\} | \bar{q}, c=+1 \rangle,$$

$$T_1^0 = (4\pi)^{1/2} f \bar{\mu}^{-1} i$$

$$\langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \pi_1^+ \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{V}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_1} + \right.$$

$$\left. (2M)^{-1} (\bar{p}_1' \pi_1^+ \frac{(-i)}{\sqrt{V}} \sum_x \sqrt{\frac{x_0}{2}} a_x^- e^{i\bar{x} \cdot \bar{x}_1} + \pi_1^+ \frac{(-i)}{\sqrt{V}} \sum_x \sqrt{\frac{x_0}{2}} a_x^- e^{i\bar{x} \cdot \bar{x}_1} \bar{p}_1) \right\} | \bar{q}, c=+1 \rangle,$$

or

$$T_1^0 = -(4\pi)^{1/2} f \bar{\mu}^{-1} \sqrt{\frac{1}{2q_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_1} \bar{\sigma}_1 \cdot$$

$$\left\{ \bar{q} - \frac{q_0}{2M} (\bar{p}_1' + \bar{p}_1) \right\}, \quad (\text{I-8})$$

once the indicated meson operations have been carried through.

Conservation of momentum at the absorption vertex gives the relation,

$$\bar{p}_1 = \bar{p}_1' - \bar{q}, \quad (\text{I-9})$$

which allows the elimination of either  $\bar{p}_1$  or  $\bar{p}_1'$  in Eq. (I-8). As stated in Chapter 2, the nucleon coordinates can be written in terms of center-of-mass and relative coordinates, according to

$$\bar{x}_{1,2} = \bar{R} \pm \frac{1}{2} \bar{r}. \quad (\text{I-10})$$

The corresponding momentum operators are

$$\bar{P}_{1,2} = \frac{1}{2} \bar{K} \pm \bar{k}. \quad (\text{I-11})$$

We desire to minimize the number of final integrals which must be done, and note that  $\bar{K}'$  vanishes in the overall-center-of-mass frame, so that

$$\bar{P}'_{1,2} = \pm \bar{k}'. \quad (\text{I-12})$$

Therefore, a good choice is to retain  $\bar{P}'_1$  in Eq. (I-8) and ultimately to evaluate the momentum operator on the final proton-proton state. Substitution of Eq. (I-9) into Eq. (I-8) then gives

$$T_1^o = -(4\pi)^{1/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_1} \gamma_1^+ \sigma_1 \cdot \left\{ \left(1 + \frac{g_0}{2M}\right) \bar{q} - \frac{g_0}{M} \bar{P}'_1 \right\}. \quad (\text{I-13})$$

The transition operator for direct absorption on nucleon (2) is obtained through the substitution, (1)  $\leftrightarrow$  (2), in Eq. (I-13). The result is

$$T_2^o = -(4\pi)^{1/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} \gamma_2^+ \sigma_2 \cdot \left\{ \left(1 + \frac{g_0}{2M}\right) \bar{q} - \frac{g_0}{M} \bar{P}'_2 \right\}. \quad (\text{I-14})$$

Finally, the full transition operator is obtained by adding Eqs. (I-13) and (I-14) and making use of the transformations (I-10) and (I-12). The nuclear transition operator for direct absorption is then, upon rearrangement, given by

$$\begin{aligned}
 T^o = & -(4\pi)^{\frac{1}{2}} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{R}} \\
 & \left\{ (e^{i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_1^+ \bar{\sigma}_1 + e^{-i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_2^+ \bar{\sigma}_2) \cdot \left(1 + \frac{g_0}{2M}\right) \bar{q} \right. \\
 & \left. - (e^{i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_1^+ \bar{\sigma}_1 - e^{-i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_2^+ \bar{\sigma}_2) \cdot \left(\frac{g_0}{M}\right) \bar{k}' \right\}. \tag{I-15}
 \end{aligned}$$

The transition operator involving the first s-wave scattering Hamiltonian,  $H^1$ , is designated  $T^1$ . We first consider the process for scattering on nucleon (2), forward propagation in time for the intermediate meson and absorption on nucleon (1). This process is represented by the graph in Fig. 25; the corresponding operator is labeled  $T_{12}^1$ .

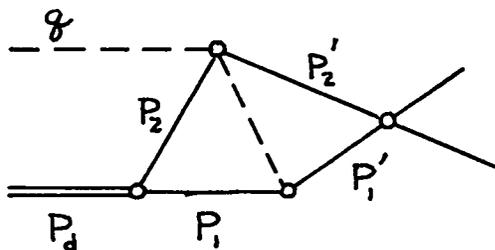


Fig. 25. Graph for absorption on nucleon (1) preceded by scattering on nucleon (2).

According to Eq. (2-1), this operator may be written as

$$T'_{12+} = \langle 0 | H_1^0 (E - H + i\epsilon)^{-1} H_2' | \bar{q}, c=+1 \rangle, \quad (\text{I-16})$$

or, making use of the first expression in Eq. (2-3), as

$$\begin{aligned} T'_{12+} &= \langle 0 | (4\pi)^{1/2} \int \bar{\mu}' i \\ &\bar{\sigma}_i \cdot \{ \bar{\nabla}_\pi \tau_i \cdot \phi_i + (2M)^{-1} (\bar{P}_i \tau_i \cdot \pi_i + \tau_i \cdot \pi_i \bar{P}_i) \} \\ &(E - H + i\epsilon)^{-1} 4\pi \lambda_i \bar{\mu}' \phi_2 \cdot \phi_2 | \bar{q}, c=+1 \rangle. \end{aligned} \quad (\text{I-17})$$

The expansion of  $\phi \cdot \phi$ , by analogy with that of  $\tau \cdot \phi$  in Eq. (I-7), is given by

$$\phi \cdot \phi = \phi^+ \phi^- + \phi^- \phi^+ + \phi^3 \phi^3. \quad (\text{I-18})$$

In the matrix element, the  $\phi^3 \phi^3$  term represents a self-energy effect and should be omitted. The remaining creation and destruction operators couple as follows:

$$\begin{aligned} \phi^+ \phi^- + \phi^- \phi^+ &\sim \\ a^+ \bar{a} + a^+ b^+ + b^- \bar{a} + b^- b^+ + \bar{a} a^+ + \bar{a} b^- + b^+ a^+ + b^+ b^-. \end{aligned} \quad (\text{I-19})$$

In operating upon the initial state with Eq. (I-19), the third, sixth and eighth terms vanish, the second and seventh lead to three-meson states and the fourth represents a self-energy term. Therefore, only the first and the fifth terms can contribute here, so that we may write

$$\begin{aligned} \phi_2 \cdot \phi_2 | \bar{q}, c=+1 \rangle = & \left\{ \frac{1}{\sqrt{V}} \sum_{x'} \sqrt{\frac{1}{2x'_0}} a_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{1}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \right. \\ & \left. + \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{1}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \frac{1}{\sqrt{V}} \sum_{x'} \sqrt{\frac{1}{2x'_0}} a_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \right\} | \bar{q}, c=+1 \rangle. \end{aligned} \quad (\text{I-20})$$

By dropping one self-energy term and a Bose term, this result can be written as

$$\phi_2 \cdot \phi_2 | \bar{q}, c=+1 \rangle = 2 \sqrt{\frac{1}{2q_0}} e^{i\bar{q} \cdot \bar{x}_2} \frac{1}{V} \sum_{x'} \sqrt{\frac{1}{2x'_0}} e^{-i\bar{x}' \cdot x_2} | \bar{x}', c=+1 \rangle. \quad (\text{I-21})$$

The propagator for the intermediate state is obtained by evaluating initial- and intermediate-state Hamiltonians with reference to Fig. 25. The incident-pion energy,  $q_0$ , must be equally shared by the two nucleons. If the binding energy of the deuteron and nucleon mass differences are neglected, then the propagator for intermediate-meson energy,  $x'_0$ , is given by

$$(E - H + i\epsilon)^{-1} | \psi \rangle = (q_0 - (\frac{q_0}{2} + x'_0))^{-1} | \psi \rangle,$$

or

$$(E-H+i\epsilon)^{-1}|\psi\rangle = \left(\frac{g_0}{2} - \chi_0'\right)^{-1}|\psi\rangle. \quad (\text{I-22})$$

Since the intermediate state contains a positive meson, only those terms in  $H_1^0$  which are linear in  $\bar{a}^-$  can contribute. The transition operator is therefore given by

$$\begin{aligned} T'_{12+} = & \lambda_1 (4\pi)^{3/2} \mu^{-2} f i \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \gamma_1^+ \frac{1}{\sqrt{V}} \sum_{\vec{q}} \sqrt{\frac{1}{2g_0'}} \bar{a}_{\vec{q}'} e^{i\vec{q}' \cdot \vec{x}_1} \right. \\ & \left. + (2M)^{-1} (\bar{p}'_1 + \bar{p}_1) \gamma_1^+ \frac{(-i)}{\sqrt{V}} \sum_{\vec{q}} \sqrt{\frac{g_0'}{2}} \bar{a}_{\vec{q}'} e^{i\vec{q}' \cdot \vec{x}_1} \right\} \\ & \sum_{\vec{x}'} \left(\frac{g_0}{2} - \chi_0'\right)^{-1} 2\sqrt{\frac{1}{2g_0}} e^{i\vec{q}' \cdot \vec{x}_2} \frac{1}{V} \sqrt{\frac{1}{2\chi_0'}} e^{-i\vec{x}' \cdot \vec{x}_2} |\vec{x}', c=+1\rangle. \end{aligned} \quad (\text{I-23})$$

Upon completion of the meson operations and use of Eq. (I-9), the final result becomes

$$\begin{aligned} T'_{12+} = & -\lambda_1 (4\pi)^{3/2} \mu^{-2} f \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\vec{q}' \cdot \vec{x}_2} \gamma_1^+ \bar{\sigma}_1 \cdot \\ & \frac{\int d\vec{q}' e^{i\vec{q}' \cdot (\vec{x}_1 - \vec{x}_2)} \left\{ \left(1 + \frac{g_0'}{2M}\right) \vec{q}' - \frac{g_0'}{M} \bar{p}'_1 \right\}}{(2\pi)^3 g_0' \left(\frac{g_0}{2} - g_0'\right)}. \end{aligned} \quad (\text{I-24})$$

In writing this expression, the sum over discrete momentum,  $q'$ , has been replaced with an integral over the continuous variable by the substitution,



$$\frac{1}{\sqrt{V}} \sum_{\vec{q}'} \rightarrow \int \frac{d\vec{q}'}{(2\pi)^3} . \quad (\text{I-25})$$

The case for backward propagation in time, as represented by the graph in Fig. 26, is easily obtained from the Lagrangian specification by considering a negative pi-meson in the intermediate state. Since  $H_2^1$  does not contain an isospin operator, only  $b^+$  terms can enter from  $H_1^0$ , and we can set

$$\phi \cdot \phi \sim \bar{a} \bar{b} + b \bar{a} . \quad (\text{I-26})$$

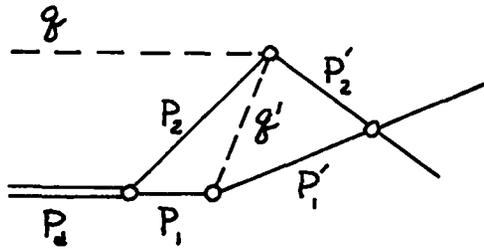


Fig. 26. Graph for absorption on nucleon (1) followed by scattering on nucleon (2).

According to our convention, the transition operator for this process is written symbolically as  $T_{12-}^1$ . It can be expressed as

$$T_{12-}^1 = \langle 0 | H_2' (E - H + i\epsilon)^{-1} H_1^0 | \bar{q}, c = +1 \rangle , \quad (\text{I-27})$$

or

$$T'_{12-} = \langle 0 | 4\pi \lambda_1 \bar{\mu}' \phi_2 \cdot \phi_2 (E-H+i\epsilon)^{-1} (4\pi)^{3/2} \{ \bar{\mu}' i \bar{\sigma}_i \cdot \{ \bar{V}_\pi \gamma_i \cdot \phi_i + (2M)^{-1} (\bar{P}_i \gamma_i \cdot \pi_i + \gamma_i \cdot \pi_i \bar{P}_i) \} | \bar{q}, c=+1 \rangle. \quad (\text{I-28})$$

The propagation factor in this case is given by

$$(E-H+i\epsilon)^{-1} |\psi\rangle = (q_0 - (q_0 + x'_0 + \frac{q_0}{2}))^{-1} |\psi\rangle,$$

or

$$(E-H+i\epsilon)^{-1} |\psi\rangle = (-\frac{q_0}{2} - x'_0)^{-1} |\psi\rangle. \quad (\text{I-29})$$

Through use of this expression and from considerations discussed above, Eq. (I-28) may be written as

$$T'_{12-} = \lambda_1 (4\pi)^{3/2} \{ \bar{\mu}'^2 i \langle 0 | \left\{ \frac{1}{\sqrt{V}} \sum_{q'} \sqrt{\frac{\Gamma}{2q'_0}} b_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{\Gamma}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} + \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{\Gamma}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \frac{1}{\sqrt{V}} \sum_{q'} \sqrt{\frac{\Gamma}{2q'_0}} b_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \right\} \sum_{x'} (-\frac{q_0}{2} - x'_0)^{-1} \bar{\sigma}_i \cdot \left\{ \bar{V}_\pi \gamma_i \frac{1}{\sqrt{V}} \sqrt{\frac{\Gamma}{2x'_0}} b_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} + (2M)^{-1} (\bar{P}'_i + \bar{P}_i) \gamma_i \frac{i}{\sqrt{V}} \sqrt{\frac{x'_0}{2}} b_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} \right\} | \bar{q}, c=+1 \rangle. \quad (\text{I-30})$$

However, conservation of momentum at the absorption vertex is now expressed by

$$\bar{p}_1 = \bar{p}'_1 + \bar{q}, \quad (\text{I-31})$$

so that the operator for the process depicted in Fig. 26 becomes

$$T'_{12-} = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2q_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} \tau_1^+ \sigma_1 \cdot \int \frac{d\bar{q}'}{(2\pi)^3} \frac{e^{-i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left(1 - \frac{q'_0}{2M}\right) \bar{q}' - \frac{q'_0}{M} \bar{p}'_1 \right\}}{q'_0 \left(\frac{q'_0}{2} + q_0\right)}. \quad (\text{I-32})$$

This can be put into a form similar to that of Eq. (I-24) by substituting  $\bar{q}' \rightarrow -\bar{q}'$  for the variable of integration. The transition operator is then given by

$$T'_{12-} = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2q_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} \tau_1^+ \sigma_1 \cdot \int \frac{d\bar{q}'}{(2\pi)^3} \frac{e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ -\left(1 - \frac{q'_0}{2M}\right) \bar{q}' - \frac{q'_0}{M} \bar{p}'_1 \right\}}{q'_0 \left(\frac{q'_0}{2} + q_0\right)}. \quad (\text{I-33})$$

We next add the graphs shown in Figs. 25 and 26 to obtain the transition operator for scattering on nucleon (2) through  $H^1$  and absorption on nucleon (1). Equation (I-24) is thus added to Eq. (I-33) to obtain

$$T'_{12} = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} \tau_1^+ \bar{\sigma}_1 \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{p}' \right\}}{(2\pi)^3 \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} . \quad (\text{I-34})$$

The nuclear transition operator for scattering on nucleon (1) and absorption on nucleon (2) is obtained through the substitutions, (1)  $\leftrightarrow$  (2) and  $\bar{q}' \rightarrow -\bar{q}'$ , in Eq. (I-34), yielding

$$T'_{21} = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_1} \tau_2^+ \bar{\sigma}_2 \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ -\left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{p}' \right\}}{(2\pi)^3 \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} . \quad (\text{I-35})$$

Finally, we add Eqs. (I-34) and (I-35) to obtain the total nuclear transition operator involving scattering through the Hamiltonian term,  $H^1$ . In writing this expression, use has also been made of the coordinate transformations (I-10) and (I-12). The final result is

$$T' = -\lambda_1 (4\pi)^{3/2} f \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{R}} \left( e^{-i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_1^+ \bar{\sigma}_1 - e^{+i\frac{\bar{q} \cdot \bar{r}}{2}} \tau_2^+ \bar{\sigma}_2 \right) \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot \bar{r}} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\}}{(2\pi)^3 \left(\frac{g_0^2}{4} - q_0'^2\right)} . \quad (\text{I-36})$$

We turn now to the transition operator involving the s-wave scattering term,  $H^2$ , and again consider the process involving scattering on nucleon (2), forward time propagation of the intermediate meson and absorption on nucleon (1). Then the graph in Fig. 25 also represents this process. The transition operator is given by

$$T_{12+}^2 = \langle 0 | H_1^0 (E - H + i\epsilon)^{-1} H_2^2 | \bar{p}, c = +1 \rangle, \quad (\text{I-37})$$

or, making use of Eq. (2-2) and the second expression in Eq. (2-3), by

$$\begin{aligned} T_{12+}^2 = & \langle 0 | (4\pi)^{1/2} \int \bar{\mu}^{-1} i \\ & \bar{\sigma}_i \cdot \left\{ \bar{v}_\pi \tau_i \cdot \phi_i + (2M)^{-1} (\bar{p}_i \tau_i \cdot \pi_i + \tau_i \cdot \pi_i \bar{p}_i) \right\} \\ & (E - H + i\epsilon)^{-1} 4\pi \lambda_2 \bar{\mu}^{-2} \tau_2 \cdot \phi_2 \times \pi_2 | \bar{p}, c = +1 \rangle. \end{aligned} \quad (\text{I-38})$$

The isospin-field product in  $H^2$  can be written as

$$\tau \cdot \phi \times \pi = \tau^1 (\phi^2 \pi^3 - \phi^3 \pi^2) + \tau^2 (\phi^3 \pi^1 - \phi^1 \pi^3) + \tau^3 (\phi^1 \pi^2 - \phi^2 \pi^1),$$

or, in terms of tensor-like components, as

$$\tau \cdot \phi \times \pi = i \left\{ \tau^+ (\phi^- \pi^3 - \phi^3 \pi^-) + \tau^- (\phi^3 \pi^+ - \phi^+ \pi^3) + \tau^3 (\phi^+ \pi^- - \phi^- \pi^+) \right\}. \quad (\text{I-39})$$

On the other hand, terms in the operator,  $H^0$ , go as the expression in Eq. (I-7). Thus, matrix elements involving the third and fourth terms ( $\tau_2^-$ ) in Eq. (I-39) vanish, as do those of the form of the second term ( $\tau_1^-$ ) in Eq. (I-7). The operator,  $T_{12+}^2$ , can therefore be written as

$$\begin{aligned}
T_{12+}^2 = & -\lambda_2 (4\pi)^{3/2} f \bar{\mu}^{-3} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \tau_1^+ \phi_1^- + (2M)^- (\bar{p}_1^+ \tau_1^+ \bar{p}_1^- + \tau_1^+ \pi_1^+ \bar{p}_1^-) \right\} \\
& (E - H + i\epsilon)^- \tau_2^3 (\phi_2^+ \pi_2^- - \phi_2^- \pi_2^+) | \bar{q}, c = +1 \rangle \\
& - \lambda_2 (4\pi)^{3/2} f \bar{\mu}^{-3} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \tau_1^3 \phi_1^3 + (2M)^- (\bar{p}_1^3 \tau_1^3 \bar{p}_1^3 + \tau_1^3 \pi_1^3 \bar{p}_1^-) \right\} \\
& (E - H + i\epsilon)^- \tau_2^+ (\phi_2^- \pi_2^3 - \phi_2^3 \pi_2^-) | \bar{q}, c = +1 \rangle, \tag{I-40}
\end{aligned}$$

or

$$\begin{aligned}
T_{12+}^2 = & -\lambda_2 (4\pi)^{3/2} f \bar{\mu}^{-3} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \tau_1^+ \frac{1}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{1}{2\bar{q}'_0}} a_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \right. \\
& + (2M)^- (\bar{p}_1^+ + \bar{p}_1^-) \tau_1^+ \frac{(-i)}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{\bar{q}'_0}{2}} a_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \left. \right\} \sum_{x'} (\frac{\bar{q}'_0}{2} - x'_0)^- \\
& \tau_2^3 \left\{ \frac{1}{\sqrt{V}} \sqrt{\frac{1}{2x'_0}} a_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_x \sqrt{\frac{x'_0}{2}} a_x^- e^{i\bar{x}' \cdot \bar{x}_2} \right. \\
& \left. - \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{1}{2x'_0}} a_x^- e^{i\bar{x}' \cdot \bar{x}_2} \frac{i}{\sqrt{V}} \sqrt{\frac{x'_0}{2}} a_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \right\} | \bar{q}, c = +1 \rangle \\
& - \lambda_2 (4\pi)^{3/2} f \bar{\mu}^{-3} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \tau_1^3 \frac{1}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{1}{2\bar{q}'_0}} c_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \right.
\end{aligned}$$

$$\begin{aligned}
& + (2M)'(\bar{P}' + \bar{P}) \gamma_1^3 \frac{(-i)}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{q_0'}{2}} c_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \left\{ \sum_{\bar{x}'} \left( \frac{q_0'}{2} - x_0' \right)^{-1} \right. \\
& \gamma_2^+ \left\{ \frac{1}{\sqrt{V}} \sum_{\bar{x}} \sqrt{\frac{1}{2x_0}} a_{\bar{x}} e^{i\bar{x} \cdot \bar{x}_2} \frac{i}{\sqrt{V}} \sqrt{\frac{x_0'}{2}} c_{\bar{x}'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \right. \\
& \left. \left. - \frac{1}{\sqrt{V}} \sqrt{\frac{1}{2x_0'}} c_{\bar{x}'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_{\bar{x}} \sqrt{\frac{x_0}{2}} a_{\bar{x}} e^{i\bar{x} \cdot \bar{x}_2} \right\} \right\} | \bar{q}, c = +1 \rangle .
\end{aligned} \tag{I-41}$$

In writing this equation, use is made of the propagator result in Eq. (I-22); expressions for the contributing field components are obtained from Eq. (I-2). Upon carrying out the indicated operations, we get the result,

$$\begin{aligned}
T_{12+}^2 & = -\lambda_2 (4\pi)^{3/2} \mu^{-3} \sqrt{\frac{1}{2q_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} (\gamma_1^+ \gamma_2^3 - \gamma_1^3 \gamma_2^+) \bar{\sigma}_1 \cdot \\
& \int \frac{d\bar{q}'}{(2\pi)^3} \frac{e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} (q_0 + q_0') \left\{ \left( 1 + \frac{q_0'}{2M} \right) \bar{q}' - \frac{q_0'}{M} \bar{P}' \right\}}{2q_0' \left( \frac{q_0}{2} - q_0' \right)} .
\end{aligned} \tag{I-42}$$

Use has also been made of the momentum relationship in Eq. (I-9).

The transition operator for backward time propagation,  $T_{12-}^2$ , is obtained by arguments similar to those used in the derivation of  $T_{12+}^2$ . The operator is defined by

$$T_{12-}^2 = \langle 0 | H_2^2 (E - H + i\epsilon)^{-1} H_1^0 | \bar{q}, c = +1 \rangle . \tag{I-43}$$

It is given in terms of its nonvanishing field components by

$$\begin{aligned}
T_{12-}^2 &= -\lambda_2 (4\pi)^{3/2} \int \bar{\mu}^{-3} \langle 0 | \gamma_2^* (\phi_2^- \pi_2^3 - \phi_2^3 \pi_2^-) (E-H+i\epsilon)^{-1} \\
&\bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \gamma_1^3 \phi_1^3 + (2M)^{-1} (\bar{P}_1 \gamma_1^3 \pi_1^3 + \gamma_1^3 \pi_1^3 \bar{P}_1) \right\} | \bar{q}, c=+1 \rangle \\
&- \lambda_2 (4\pi)^{3/2} \int \bar{\mu}^{-3} \langle 0 | \gamma_2^3 (\phi_2^+ \pi_2^- - \phi_2^- \pi_2^+) (E-H+i\epsilon)^{-1} \\
&\bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \gamma_1^+ \phi_1^- + (2M)^{-1} (\bar{P}_1 \gamma_1^+ \pi_1^- + \gamma_1^+ \pi_1^- \bar{P}_1) \right\} | \bar{q}, c=+1 \rangle. \quad (I-44)
\end{aligned}$$

With use of the appropriate terms of these field components and the propagator result given in Eq. (I-29), the transition operator becomes

$$\begin{aligned}
T_{12-}^2 &= \\
&- \lambda_2 (4\pi)^{3/2} \int \bar{\mu}^{-3} \langle 0 | \gamma_2^* \left\{ \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{1}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_{q'} \sqrt{\frac{q_0'}{2}} c_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \right. \\
&- \left. \frac{1}{\sqrt{V}} \sum_{q'} \sqrt{\frac{1}{2q_0'}} c_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_x \sqrt{\frac{x_0}{2}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \right\} \\
&\sum_{x'} \left( -\frac{q_0}{2} - x_0' \right)^{-1} \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \gamma_1^3 \frac{1}{\sqrt{V}} \sqrt{\frac{1}{2x_0'}} c_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} \right. \\
&+ \left. (2M)^{-1} (\bar{P}_1' + \bar{P}_1) \gamma_1^3 \frac{i}{\sqrt{V}} \sqrt{\frac{x_0'}{2}} c_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} \right\} | \bar{q}, c=+1 \rangle \\
&- \lambda_2 (4\pi)^{3/2} \int \bar{\mu}^{-3} \langle 0 | \gamma_2^3 \left\{ \frac{1}{\sqrt{V}} \sum_{q'} \sqrt{\frac{1}{2q_0'}} b_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_x \sqrt{\frac{x_0}{2}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \right. \\
&- \left. \frac{1}{\sqrt{V}} \sum_x \sqrt{\frac{1}{2x_0}} a_x^- e^{i\bar{x} \cdot \bar{x}_2} \frac{(-i)}{\sqrt{V}} \sum_{q'} \sqrt{\frac{q_0'}{2}} b_{q'}^- e^{i\bar{q}' \cdot \bar{x}_2} \right\} \\
&\sum_{x'} \left( -\frac{q_0}{2} - x_0' \right)^{-1} \bar{\sigma}_1 \cdot \left\{ \bar{\nabla}_\pi \gamma_1^+ \frac{1}{\sqrt{V}} \sqrt{\frac{1}{2x_0'}} b_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} \right. \\
&+ \left. (2M)^{-1} (\bar{P}_1' + \bar{P}_1) \gamma_1^+ \frac{i}{\sqrt{V}} \sqrt{\frac{x_0'}{2}} b_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_1} \right\} | \bar{q}, c=+1 \rangle. \quad (I-45)
\end{aligned}$$



After performing the indicated meson operations, and making use of the momentum relation of Eq. (I-31) and the substitution,  $\bar{q}' \rightarrow -\bar{q}'$ , this operator can be written as

$$T_{12-}^2 = -\lambda_2 (4\pi)^{3/2} f \mu^{-3} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}' \cdot \bar{x}_2} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_1 \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} (g_0 - g_0') \left\{ -\left(1 - \frac{g_0'}{2M}\right) \bar{q}' - \frac{g_0'}{M} \bar{p}'_1 \right\}}{(2\pi)^3 2g_0' \left(\frac{g_0}{2} + g_0'\right)} . \quad (\text{I-46})$$

The full transition operator for scattering on nucleon (2) through  $H^2$  and absorption on nucleon (1) is obtained by adding Eqs. (I-42) and (I-46). The result is

$$T_{12}^2 = -\lambda_2 (4\pi)^{3/2} f \frac{3g_0}{4\mu} \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}' \cdot \bar{x}_2} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_1 \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left(2 + \frac{g_0^2 + 2g_0'^2}{3g_0 M}\right) \bar{q}' - \frac{2(g_0^2 + 2g_0'^2)}{3g_0 M} \bar{p}'_1 \right\}}{(2\pi)^3 \left(\frac{g_0}{2} - g_0'\right) \left(\frac{g_0}{2} + g_0'\right)} . \quad (\text{I-47})$$

The case for scattering on nucleon (1) and absorption on nucleon (2) is obtained through the substitutions, (1)  $\leftrightarrow$  (2), and  $\bar{q}' \rightarrow -\bar{q}'$ . This result is given by

$$T_{21}^2 = -\lambda_2 (4\pi)^{3/2} f \frac{3g_0}{4\mu} \mu^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}' \cdot \bar{x}_1} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_2 \cdot \int \frac{d\bar{q}' e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left(2 + \frac{g_0^2 + 2g_0'^2}{3g_0 M}\right) \bar{q}' + \frac{2(g_0^2 + 2g_0'^2)}{3g_0 M} \bar{p}'_2 \right\}}{(2\pi)^3 \left(\frac{g_0}{2} - g_0'\right) \left(\frac{g_0}{2} + g_0'\right)} . \quad (\text{I-48})$$

The total transition operator involving the Hamiltonian term,  $H^2$ , is obtained by adding Eqs. (I-47) and (I-48). Use is also made of the coordinate relations in Eqs. (I-10) and (I-12) to gain the result,

$$\begin{aligned}
T^2 = & -\lambda_2 (4\pi)^{3/2} f \bar{\mu}^{-2} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}} \\
& \frac{3g_0}{4\mu} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 + e^{+i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \right) \cdot \\
& \int \frac{d\bar{q}'}{(2\pi)^3} \frac{e^{i\bar{q}'\cdot\bar{r}} \left\{ \left( 2 + \frac{q_0^2 + 2g_0^2}{3g_0 M} \right) \bar{q}' - \frac{2(q_0^2 + 2g_0^2)}{3g_0 M} \bar{k}' \right\}}{\left( \frac{q_0^2}{4} - q_0^2 \right)} . \quad (I-49)
\end{aligned}$$

We next consider the transition operator involving the first p-wave scattering Hamiltonian given in Eq. (2-5). The dot product in  $H^3$  refers to the fields as well as the pion momenta, and therefore the derivation of  $T^3$  follows closely that of  $T^1$ . We consider the process for scattering on nucleon (2) and absorption on nucleon (1) with forward time propagation, and Fig. 25 is once again applicable. This operator is given by

$$T_{12+}^3 = \langle 0 | H_1^0 (E - H + i\epsilon)^{-1} H_2^3 | \bar{q}, c = +1 \rangle , \quad (I-50)$$

or

$$\begin{aligned}
T_{12+}^3 = & \langle 0 | (4\pi)^{1/2} f \bar{\mu}^{-1} i \bar{\sigma}_1 \cdot \left\{ i \bar{q}_\pi \tau_1 \cdot \phi_1 + (2M)^{-1} (\bar{p}_1 \tau_1 \cdot \pi_1 + \tau_1 \cdot \pi_1 \bar{p}_1) \right\} \\
& (E - H + i\epsilon)^{-1} (-4\pi) \lambda_3 \bar{\mu}^{-3} (\alpha^2 + q^2)^2 \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_2 \right) \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_2 \right) | \bar{q}, c = +1 \rangle . \quad (I-51)
\end{aligned}$$

By analogy with the derivation of  $T_{12+}^1$ , this operator is easily shown to be

$$\begin{aligned}
 T_{12+}^3 &= \lambda_3 (4\pi)^{3/2} \int \mu^{-4} i \langle 0 | \bar{\sigma}_1 \cdot \left\{ \frac{1}{\sqrt{V}} \gamma_1^+ \frac{1}{\sqrt{V}} \sum_{\vec{q}'} \sqrt{\frac{1}{2q_0'}} a_{\vec{q}'}^- e^{i\vec{q}' \cdot \vec{x}_1} \right. \\
 &+ (2M)^{-1} (\bar{p}_1' + \bar{p}_1) \gamma_1^+ \frac{(-i)}{\sqrt{V}} \sum_{\vec{q}'} \sqrt{\frac{q_0'}{2}} a_{\vec{q}'}^- e^{i\vec{q}' \cdot \vec{x}_1} \left. \right\} \\
 &\sum_{\vec{x}_1'} \left( \frac{q_0'}{2} - x_0' \right)^{-1} \frac{2}{V} \sqrt{\frac{1}{2q_0'}} e^{i\vec{q}' \cdot \vec{x}_2} \frac{\alpha^2 + q_0'^2}{\alpha^2 + x_0'^2} \sqrt{\frac{1}{2x_0'}} \bar{x}_1 \cdot \vec{q}' e^{-i\vec{x}_1' \cdot \vec{x}_2} | \vec{x}_1', c = +1 \rangle,
 \end{aligned} \tag{I-52}$$

or, in final form,

$$\begin{aligned}
 T_{12+}^3 &= -\lambda_3 (4\pi)^{3/2} \int \mu^{-4} \sqrt{\frac{1}{2q_0'}} \frac{1}{\sqrt{V}} e^{i\vec{q}' \cdot \vec{x}_2} \gamma_1^+ \bar{\sigma}_1 \cdot \\
 &\frac{\int d\vec{q}' (\alpha^2 + q_0'^2) e^{i\vec{q}' \cdot (\vec{x}_1 - \vec{x}_2)} \left\{ \left( 1 + \frac{q_0'}{2M} \right) \vec{q}' - \frac{q_0'}{M} \bar{p}_1' \right\} \vec{q}' \cdot \vec{q}}{(2\pi)^3 (\alpha^2 + q_0'^2) q_0' \left( \frac{q_0'}{2} - q_0' \right)}. \tag{I-53}
 \end{aligned}$$

The operator for backward propagation, as represented by the graph of Fig. 26, can be obtained by inspection. The result is

$$\begin{aligned}
 T_{12-}^3 &= -\lambda_3 (4\pi)^{3/2} \int \mu^{-4} \sqrt{\frac{1}{2q_0'}} \frac{1}{\sqrt{V}} e^{i\vec{q}' \cdot \vec{x}_2} \gamma_1^+ \bar{\sigma}_1 \cdot \\
 &\frac{\int d\vec{q}' (\alpha^2 + q_0'^2) e^{i\vec{q}' \cdot (\vec{x}_1 - \vec{x}_2)} \left\{ - \left( 1 - \frac{q_0'}{2M} \right) \vec{q}' - \frac{q_0'}{M} \bar{p}_1' \right\} \vec{q}' \cdot \vec{q}}{(2\pi)^3 (\alpha^2 + q_0'^2) q_0' \left( \frac{q_0'}{2} + q_0' \right)}. \tag{I-54}
 \end{aligned}$$

Following the usual procedure, Eqs. (I-53) and (I-54) are added to obtain the transition operator for scattering on nucleon (2) and absorption on nucleon (1):

$$T_{12}^3 = -\lambda_3 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\vec{q} \cdot \vec{x}_2} \gamma_1^+ \bar{\sigma}_1 \cdot \int \frac{d\vec{q}' (\alpha^2 + q'^2) e^{i\vec{q}' \cdot (\vec{x}_1 - \vec{x}_2)} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{p}' \right\} \bar{q}' \cdot \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} \quad (I-55)$$

Again through the substitutions, (1)  $\leftrightarrow$  (2) and  $\vec{q}' \rightarrow -\vec{q}'$ , we obtain the operator for scattering on nucleon (1) and absorption on nucleon (2), which is

$$T_{21}^3 = -\lambda_3 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\vec{q} \cdot \vec{x}_1} \gamma_2^+ \bar{\sigma}_2 \cdot \int \frac{d\vec{q}' (\alpha^2 + q'^2) e^{i\vec{q}' \cdot (\vec{x}_1 - \vec{x}_2)} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' + \frac{g_0}{M} \bar{p}' \right\} \bar{q}' \cdot \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} \quad (I-56)$$

Addition of Eqs. (I-55) and (I-56) then gives the full transition operator involving scattering through the p-wave term,  $H^3$ . The result is

$$T^3 = -\lambda_3 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\vec{q} \cdot \vec{R}} \left( e^{-i\frac{\vec{q} \cdot \vec{R}}{2}} \gamma_1^+ \bar{\sigma}_1 + e^{+i\frac{\vec{q} \cdot \vec{R}}{2}} \gamma_2^+ \bar{\sigma}_2 \right) \cdot \int \frac{d\vec{q}' (\alpha^2 + q'^2) e^{i\vec{q}' \cdot \vec{R}} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \vec{k}' \right\} \bar{q}' \cdot \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0^2}{4} - q_0'^2\right)} \quad (I-57)$$

Finally, the transition operator is developed which involves scattering through  $H^4$ , the second p-wave Hamiltonian in Eq. (2-5). As previously stated, the dot-cross product in  $H^4$  refers to the isospin-field vectors as well as to spin-momentum. This derivation then closely parallels that of  $T^2$ . For example, use is made of an expression similar to Eq. (I-39) to write the transition operator defined by

$$T_{12+}^4 = \langle 0 | H_1^0 (E-H+i\epsilon)^{-1} H_2^4 | \bar{q}, C=+1 \rangle \quad (\text{I-58})$$

as

$$\begin{aligned} T_{12+}^4 = & -\lambda_4 (4\pi)^{3/2} \int \mu^{-4} \langle 0 | \bar{\sigma}_1 \cdot \left\{ i \bar{q}_1 \gamma_1^+ \phi_1^- + (2M)^{-1} (\bar{P}_1 \gamma_1^+ \pi_1^- + \gamma_1^+ \pi_1^- \bar{P}_1) \right\} \\ & (E-H+i\epsilon)^{-1} \bar{\sigma}_2 \cdot \gamma_2^3 (\alpha^2 + q^2)^2 \\ & \left\{ - \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^+ \right) \times \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^- \right) + \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^- \right) \times \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^+ \right) \right\} | \bar{q}, C=+1 \rangle \\ & - \lambda_4 (4\pi)^{3/2} \int \mu^{-4} \langle 0 | \bar{\sigma}_1 \cdot \left\{ i \bar{q}_1 \gamma_1^3 \phi_1^3 + (2M)^{-1} (\bar{P}_1 \gamma_1^3 \pi_1^3 + \gamma_1^3 \pi_1^3 \bar{P}_1) \right\} \\ & (E-H+i\epsilon)^{-1} \bar{\sigma}_2 \cdot \gamma_2^+ (\alpha^2 + q^2)^2 \\ & \left\{ - \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^- \right) \times \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^3 \right) + \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^3 \right) \times \left( \frac{\bar{q}_\pi}{\alpha^2 + q_\pi^2} \phi_{1/2}^- \right) \right\} | \bar{q}, C=+1 \rangle. \end{aligned} \quad (\text{I-59})$$

This transition operator is given fully in potentially contributing terms by

$$\begin{aligned}
T_{12+}^4 = & -\lambda_4 (4\pi)^{3/2} \int \bar{\mu}^{-4} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nu}_\pi \bar{\nu}_1^+ \frac{1}{\sqrt{V}} \sum_{\bar{q}'} \frac{1}{\sqrt{2\bar{q}'_0}} \bar{a}_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \right. \\
& + (2M)^{-1} (\bar{P}'_1 + \bar{P}) \bar{\nu}_1^+ \frac{(-i)}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{\bar{q}'_0}{2}} \bar{a}_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \left. \right\} \sum_{x'} \left( \frac{\bar{q}'_0}{2} - x'_0 \right)^{-1} \\
& (\alpha^2 + \bar{q}^2)^2 \bar{\sigma}_2 \cdot \bar{\nu}_2^3 \left\{ \frac{\bar{x}'_1}{\alpha^2 + x'^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x'_0}} \bar{a}_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \times \sum_x \frac{\bar{x}}{\alpha^2 + x^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x_0}} \bar{a}_x^- e^{i\bar{x} \cdot \bar{x}_2} \right. \\
& - \left. \sum_x \frac{\bar{x}}{\alpha^2 + x^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x_0}} \bar{a}_x^- e^{i\bar{x} \cdot \bar{x}_2} \times \frac{\bar{x}'_1}{\alpha^2 + x'^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x'_0}} \bar{a}_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \right\} | \bar{q}, c=+1 \rangle \\
& -\lambda_4 (4\pi)^{3/2} \int \bar{\mu}^{-4} \langle 0 | \bar{\sigma}_1 \cdot \left\{ \bar{\nu}_\pi \bar{\nu}_1^3 \frac{1}{\sqrt{V}} \sum_{\bar{q}'} \frac{1}{\sqrt{2\bar{q}'_0}} \bar{c}_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \right. \\
& + (2M)^{-1} (\bar{P}'_1 + \bar{P}) \bar{\nu}_1^3 \frac{(-i)}{\sqrt{V}} \sum_{\bar{q}'} \sqrt{\frac{\bar{q}'_0}{2}} \bar{c}_{\bar{q}'}^- e^{i\bar{q}' \cdot \bar{x}_1} \left. \right\} \sum_{x'} \left( \frac{\bar{q}'_0}{2} - x'_0 \right)^{-1} \\
& (\alpha^2 + \bar{q}^2)^2 \bar{\sigma}_2 \cdot \bar{\nu}_2^+ \left\{ \sum_x \frac{\bar{x}}{\alpha^2 + x^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x_0}} \bar{a}_x^- e^{i\bar{x} \cdot \bar{x}_2} \times \frac{\bar{x}'_1}{\alpha^2 + x'^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x'_0}} \bar{c}_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \right. \\
& - \left. \frac{\bar{x}'_1}{\alpha^2 + x'^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x'_0}} \bar{c}_{x'}^+ e^{-i\bar{x}' \cdot \bar{x}_2} \times \sum_x \frac{\bar{x}}{\alpha^2 + x^2} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2x_0}} \bar{a}_x^- e^{i\bar{x} \cdot \bar{x}_2} \right\} | \bar{q}, c=+1 \rangle.
\end{aligned}$$

(I-60)

The indicated operations are carried through to obtain

$$\begin{aligned}
T_{12+}^4 = & -\lambda_4 (4\pi)^{3/2} \int \bar{\mu}^{-4} \frac{1}{\sqrt{2\bar{q}'_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}' \cdot \bar{x}_2} (\bar{\nu}_1^+ \bar{\nu}_2^3 - \bar{\nu}_1^3 \bar{\nu}_2^+) \bar{\sigma}_1 \cdot \\
& \frac{\int d\bar{q}' (\alpha^2 + \bar{q}'^2) e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left( 1 + \frac{\bar{q}'_0}{2M} \right) \bar{q}' - \frac{\bar{q}'_0}{M} \bar{P}'_1 \right\} i \bar{\sigma}_2 \cdot \bar{q}' \times \bar{q}}{(2\pi)^3 (\alpha^2 + \bar{q}'^2) \bar{q}'_0 \left( \frac{\bar{q}'_0}{2} - \bar{q}'_0 \right)}.
\end{aligned}$$

(I-61)

Using techniques similar to those previously employed, the backward scattering result can be shown to be

$$T_{12-}^4 = -\lambda_4 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_1 \cdot$$

$$\frac{\int d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ -\left(1 - \frac{g_0'}{2M}\right) \bar{q}' - \frac{g_0'}{M} \bar{p}'_1 \right\} i \bar{\sigma}_2 \cdot \bar{q}' \times \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \frac{g_0'}{b_0} \left(\frac{g_0}{2} + q'_0\right)} .$$

(I-62)

Once again, the forward and backward propagation terms are added to obtain the operator for scattering on nucleon (2) and absorption on nucleon (1). The result is

$$T_{12}^4 = -\lambda_4 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_2} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_1 \cdot$$

$$\frac{\int d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{p}'_1 \right\} i \bar{\sigma}_2 \cdot \bar{q}' \times \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} .$$

(I-63)

We next get the transition operator for scattering on nucleon (1) and absorption on nucleon (2) through the substitutions, (1)  $\leftrightarrow$  (2) and  $\bar{q}' \rightarrow -\bar{q}'$ , yielding

$$T_{21}^4 = -\lambda_4 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q} \cdot \bar{x}_1} (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \bar{\sigma}_2 \cdot$$

$$\frac{\int d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}' \cdot (\bar{x}_1 - \bar{x}_2)} \left\{ -\left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{p}'_2 \right\} i \bar{\sigma}_1 \cdot \bar{q}' \times \bar{q}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0}{2} - q'_0\right) \left(\frac{g_0}{2} + q'_0\right)} .$$

(I-64)

Finally, the total transition operator involving scattering through the Hamiltonian term,  $H^4$ , is obtained by adding Eqs. (I-63) and (I-64). The result is written as

$$\begin{aligned}
 T^4 &= -\lambda_4 (4\pi)^{3/2} \int \bar{\mu}^{-4} \frac{\sqrt{1}}{\sqrt{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}} \\
 & (\gamma_1^+ \gamma_2^3 - \gamma_1^3 \gamma_2^+) \int \frac{d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}'\cdot\bar{r}}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{g_0^2}{4} - q_0'^2\right)} \\
 & \left\{ e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_1 \cdot \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\} i\bar{\sigma}_2 \cdot \bar{q}' \times \bar{q} \right. \\
 & \left. - e^{+i\frac{\bar{q}\cdot\bar{r}}{2}} \bar{\sigma}_2 \cdot \left\{ \left(2 + \frac{g_0}{2M}\right) \bar{q}' - \frac{g_0}{M} \bar{k}' \right\} i\bar{\sigma}_1 \cdot \bar{q}' \times \bar{q} \right\}. \quad (I-65)
 \end{aligned}$$

According to Eq. (2-1), the complete transition operator,  $T$ , is the sum of Eqs. (I-15), (I-36), (I-49), (I-57) and (I-65). However, in several calculations, certain terms will be omitted for illustrative purposes. For convenience, the required equations are repeated in Chapter 2.



## APPENDIX II

### PHENOMENOLOGICAL NUCLEON-NUCLEON POTENTIALS

The Hamada-Johnston model<sup>105</sup> is an energy-independent potential consisting of polynomials in Yukawa terms. The free parameters have been fixed phenomenologically using deuteron and scattering data. The central and tensor terms reduce to one-pion exchange (OPEP) at large radius. Prominent features include a hard core at radius,  $r_0$ , which in these calculations is taken to be 0.485345 fermis. The potential is given by

$$V = V_c + V_T S_{12} + V_{LS} \bar{L} \cdot \bar{S} + V_{LL} L_{12}, \quad (\text{II-1})$$

where the subscripts imply central, tensor, spin-orbit and quadratic spin-orbit terms. These are defined as follows:

$$V_c = 0.08 \left(\frac{1}{3}\mu\right) \bar{\tau}_1 \cdot \bar{\tau}_2 \bar{\sigma}_1 \cdot \bar{\sigma}_2 Y(\mu r) \left\{ 1 + a_c Y(\mu r) + b_c Y^2(\mu r) \right\},$$

$$V_T = 0.08 \left(\frac{1}{3}\mu\right) \bar{\tau}_1 \cdot \bar{\tau}_2 Z(\mu r) \left\{ 1 + a_T Y(\mu r) + b_T Y^2(\mu r) \right\},$$

$$V_{LS} = \mu G_{LS} Y^2(\mu r) \left\{ 1 + b_{LS} Y(\mu r) \right\}$$

and

$$V_{LL} = \mu G_{LL} (\mu r)^{-2} Z(\mu r) \left\{ 1 + a_{LL} Y(\mu r) + b_{LL} Y^2(\mu r) \right\}. \quad (\text{II-2})$$

The quantities  $Y(r)$  and  $Z(r)$  are given by

$$Y(\mu r) = \frac{e^{-\mu r}}{\mu r}$$

and

$$Z(\mu r) = \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right) \frac{e^{-\mu r}}{\mu r}. \quad (\text{II-3})$$

$\bar{L} \cdot \bar{S}$  is the usual spin-orbit product, while the operators  $L_{12}$  and  $S_{12}$  are defined by

$$L_{12} = \left\{ \delta_{LJ} + \bar{\sigma}_1 \cdot \bar{\sigma}_2 \right\} \bar{L} \cdot \bar{L} - (\bar{L} \cdot \bar{S})^2$$

and

$$S_{12} = 3(\bar{\sigma}_1 \cdot \hat{r})(\bar{\sigma}_2 \cdot \hat{r}) - \bar{\sigma}_1 \cdot \bar{\sigma}_2. \quad (\text{II-4})$$

The pion mass is taken to be  $\mu = (1.415f)^{-1}$ . The parameters determined by Hamada and Johnston are given in Table II.

Table II  
Parameters of the Hamada-Johnston Potential

State	$\underline{a_c}$	$\underline{b_c}$	$\underline{a_T}$	$\underline{b_T}$	$\underline{G_{LS}}$	$\underline{b_{LS}}$	$\underline{G_{LL}}$	$\underline{a_{LL}}$	$\underline{b_{LL}}$
singlet even	8.7	10.6					-0.000891	0.2	-0.2
triplet odd	-9.07	3.48	-1.29	0.55	0.1961	-7.12	-0.000891	-7.26	6.92
triplet even	6.0	-1.0	-0.5	0.2	0.0743	-0.1	0.00267	1.8	-0.4
singlet odd	-8.0	12.0					-0.00267	2.0	6.0

The Schrödinger equations obtained with this potential are, for the singlet and uncoupled triplet states, respectively,

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{J(J+1)}{r^2} - MV_c(r) + 2J(J+1)MV_{LL}(r) \right\} u(r) = 0$$

and

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{J(J+1)}{r^2} - MV_c(r) - 2MV_T(r) + MV_{LS}(r) - \{2J(J+1) - 1\}MV_{LL}(r) \right\} u(r) = 0 ,$$

while the equations for the coupled triplet case are given by

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{J(J-1)}{r^2} - MV_c(r) + \frac{2(J-1)}{2J+1}MV_T(r) - (J-1)MV_{LS}(r) - (J-1)MV_{LL}(r) \right\} u(r) - \frac{6\sqrt{J(J+1)}}{2J+1}MV_T(r)\omega(r) = 0$$

and

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{(J+1)(J+2)}{r^2} - MV_c(r) + \frac{2(J+2)}{2J+1}MV_T(r) + (J+2)MV_{LS}(r) + (J+2)MV_{LL}(r) \right\} \omega(r) - \frac{6\sqrt{J(J+1)}}{2J+1}MV_T(r)u(r) = 0 .$$

(II-5)

With the parameters of Table II, solution of the coupled differential equations for the deuteron using the method of Appendix III results in an energy eigenvalue of -2.2688 MeV instead of the -2.226 MeV value stated by Hamada and Johnston. This higher value is confirmed by Humberston and Wallace,<sup>106</sup> who obtained better agreement with experiment by varying either the pion mass or the hard core radius. For the absorption calculations, the original Hamada-Johnston parameters are used.

In the Boundary Condition Model (BCM),<sup>107</sup> the potential interaction is determined largely from field-theoretic forms involving one and two pion,  $\rho$ ,  $\omega$  and  $\eta$  meson exchange adiabatic local potentials beyond a

radius,  $r_0$ . Thus, beyond  $r_0$ , the interaction is represented by a local, energy-independent potential. Within  $r_0$ , the interaction involves many-particle exchange, and is strongly nonlocal. This nonlocal interaction is approximated with an energy-independent boundary condition on the logarithmic derivative of the wave function at  $r_0$ .

The Schrödinger equations for  $r > r_0$  are, for both the singlet and uncoupled triplet cases,

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{2\eta k}{r} + V_{\text{vac}} - \frac{L(L+1)}{r^2} - M \left\{ V_{ST} + \delta_{S1} (2 - 6\delta_{J0}) V_T \right\} \right\} u(r) = 0 ,$$

and for the coupled triplet case,

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{2\eta k}{r} + V_{\text{vac}} - \frac{J(J-1)}{r^2} - M \left\{ V_{ST} - \frac{2(J-1)}{2J+1} V_T \right\} \right\} u(r) - \frac{6M\sqrt{J(J+1)}}{2J+1} V_T \omega(r) = 0$$

and

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{2\eta k}{r} + V_{\text{vac}} - \frac{(J+1)(J+2)}{r^2} - M \left\{ V_{ST} - \frac{2(J+2)}{2J+1} V_T \right\} \right\} \omega(r) - \frac{6M\sqrt{J(J+1)}}{2J+1} V_T u(r) = 0 . \quad (\text{II-6})$$

The nuclear potential function is defined by

$$V = V_{ST} + S_{12} V_T = V_2 + V_4 + V_\rho + V_\omega + V_2 ,$$

where

$$V_2 = \frac{g^2}{12} \left(\frac{\mu}{M}\right)^2 \bar{\tau}_1 \bar{\tau}_2 \left\{ \bar{\sigma}_1 \cdot \bar{\sigma}_2 + S_{12} \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right) \right\} \frac{e^{-\mu r}}{r},$$

$$V_4 = \frac{(g')^4}{16} \left(\frac{\mu}{M}\right)^4 \left\{ -\frac{24\lambda^2}{\pi(\mu r)^2} \left(\frac{M}{\mu}\right)^2 K_1(2\mu r) + 12\lambda \left(\frac{M}{\mu}\right) \frac{(1+\mu r)^2}{(\mu r)^4} e^{-2\mu r} \right. \\ \left. - R_1(\mu r) - \bar{\sigma}_1 \cdot \bar{\sigma}_2 R_2(\mu r) - S_{12} R_3(\mu r) \right\} \mu,$$

$$V_\rho = \frac{3}{2} \sqrt{\bar{\tau}_1 \bar{\tau}_2} \left[ 1 + \frac{(1+2g_V)^2 (m_\rho)^2}{12} \left(\frac{m_\rho}{M}\right)^2 \right. \\ \left. \left\{ 2\bar{\sigma}_1 \cdot \bar{\sigma}_2 - \left(1 + \frac{3}{m_\rho r} + \frac{3}{(m_\rho r)^2}\right) S_{12} \right\} \right] \frac{e^{-m_\rho r}}{r},$$

$$V_\omega = \frac{9}{4} (N')^2 \left[ 1 + \frac{(1+2g_S)^2 (m_\omega)^2}{12} \left(\frac{m_\omega}{M}\right)^2 \right. \\ \left. \left\{ 2\bar{\sigma}_1 \cdot \bar{\sigma}_2 - \left(1 + \frac{3}{m_\omega r} + \frac{3}{(m_\omega r)^2}\right) S_{12} \right\} \right] \frac{e^{-m_\omega r}}{r}$$

and

$$V_7 = \frac{g_7^2}{12} \left(\frac{m_7}{M}\right)^2 \left[ \bar{\sigma}_1 \cdot \bar{\sigma}_2 + S_{12} \left(1 + \frac{3}{m_7 r} + \frac{3}{(m_7 r)^2}\right) \right] \frac{e^{-m_7 r}}{r}. \quad (\text{II-7})$$

The functions,  $R_i$ , used in  $V_4$  are defined by

$$R_1(x) = \frac{2}{\pi} \left\{ \bar{\tau}_1 \bar{\tau}_2 \left[ \left(\frac{12}{x^2} + \frac{23}{x^4}\right) K_1(2x) + \left(\frac{4}{x} + \frac{23}{x^3}\right) K_0(2x) \right] \right. \\ \left. + \xi (3 - 2\bar{\tau}_1 \bar{\tau}_2) \left[ \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4}\right) K_1(x) + \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3}\right) K_0(x) \right] e^{-x} \right\},$$

$$R_2(x) = \frac{2}{\pi} \left\{ - \left[ \left( \frac{8}{x^2} + \frac{12}{x^4} \right) K_1(2x) + \frac{12}{x^3} K_0(2x) \right] \right. \\ \left. + \frac{2}{3} \xi (3 - 2\bar{\tau}_1 \cdot \bar{\tau}_2) \left[ \left( \frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) K_1(x) + \left( \frac{1}{x^2} + \frac{1}{x^3} \right) K_0(x) \right] e^{-x} \right\}$$

and

$$R_3(x) = \frac{2}{\pi} \left\{ \left[ \left( \frac{4}{x^2} + \frac{15}{x^4} \right) K_1(2x) + \frac{12}{x^3} K_0(2x) \right] \right. \\ \left. - \frac{1}{3} \xi (3 - 2\bar{\tau}_1 \cdot \bar{\tau}_2) \left[ \left( \frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) K_1(x) + \left( \frac{1}{x^2} + \frac{1}{x^3} \right) K_0(x) \right] e^{-x} \right\} .$$

(II-8)

The boundary conditions are given, in the uncoupled case, by

$$r_0 \frac{d}{dr_0} u = f u ,$$

and in the coupled case, by the matrix equation,

$$\begin{pmatrix} r_0 \frac{d}{dr_0} u \\ r_0 \frac{d}{dr_0} w \end{pmatrix} = \begin{pmatrix} f_{J-1} & f_J \\ f_J & f_{J+1} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} .$$

(II-9)

By varying parameters in the Boundary Condition Model, Lomon and Feshbach were able to define different cases which satisfied the theoretical requirements. In these calculations we consider mainly cases 1 and 15, which correspond to deuteron D-state fractions of 4.60% and 7.55%, respectively. The parameters and constants defining these two cases are given in Table III, with the values for case 15 given in parentheses. The values of  $\mu$  and  $M$  are determined by the process considered. A natural

Table III

Parameters of the Boundary Condition Model Case 1 (15)

$g^2 = (g')^2 = 14.94$ (14.95)	$\lambda = 0.9343$ (0.9338)
$\xi = 0.745$ (0.755)	$r_0 = 0.51373 \mu^{-1}$
$\mu_p = 135.0$ MeV	$\mu_1 = 137.98$
$M(pp) = 938.2$	$\mu_0 = 139.0$
	$M(np) = 938.8$
	$M(nn) = 939.6$
$N^2 = 0.65$	$m_\rho = 765.0$ MeV
$(N')^2 = 1.3$	$m_\omega = 782.8$
$g_\eta^2 = 1$	$m_\eta = 548.7$

T = 1

Uncoupled	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3F_3$	$^1G_4$	$^3H_5$
f	1.8756	510	6.1	4.3	200	175	-4
(f)		(460)	(6.9)	(3.8)	(300)	(125)	(4)
Coupled		$^3P_2 - ^3F_2$			$^3F_4 - ^3H_4$		
$f_{J-1}$	0.352	( 0.342)			40		
$f_{J+1}$	20.9	(22.9)			-2.07		( -2.01)
$f_J$	-0.9	(-0.5)			-10.83		(-11.02)

T = 0

Uncoupled	$^1P_1$	$^3D_2$	$^1F_3$	$^3G_4$	$^1H_5$
f	-1.38	600	45	195	200
(f)	(-1.15)		(35)	(145)	
Coupled	$^3S_1 - ^3D_1$		$^3D_3 - ^3G_3$	$^3G_5 - ^3I_5$	
$f_{J-1}$	0.27057	(81.106)	10.7	(11.4)	10
$f_{J+1}$	150	( 150)	-0.4	(-0.6)	10
$f_J$	0.3	(-110)	6.4	( 6.2)	10

unit for use in the absorption study is the fermi. Thus, a single core radius,  $r_0$ , given in units of inverse pion mass, goes over into three distinct radii in fermis. Additional constants used are  $e = 1.60206 \times 10^{-19}$  coulombs,  $\hbar = 1.05443 \times 10^{-34}$  joule-seconds and  $c = 2.99793 \times 10^8$  meters per second.

The hyperbolic Bessel functions,  $K_0$  and  $K_1$ , are obtained from polynomial approximations.<sup>108</sup>

It is found that use of the stated case 15 parameters gives good agreement with scattering results, but a very poor value for the deuteron binding energy. This contrasts markedly with five other BCM cases in which excellent eigenvalues are obtained. Use of the case 1 values for  $g^2$ ,  $\lambda$  and  $\xi$  together with the case 15 boundary conditions gives the desired result; these parameters are used to calculate the case 15 deuteron used in the absorption studies.

The Boundary Condition Model at first exhibited sensitivity to the radial spacing near the core. This was found related to our neglect of the second-order term in calculating the wave function at  $r_0 + \epsilon$  prior to using the integration technique described in Appendix III; that is, the wave function at  $r_0 + \epsilon$  can be expressed as a Taylor series, according to

$$u(r_0 + \epsilon) = u(r_0) + \epsilon \frac{d}{dr} u(r_0) + \frac{\epsilon^2}{2!} \frac{d^2}{dr^2} u(r_0) + \dots \quad (\text{II-10})$$

The first derivative is given exactly from the boundary condition. However, the second derivative is also given exactly in this case from the



Schrödinger equation. Inclusion of this second-derivative term in the calculation significantly reduces sensitivity to the radial spacing.

For the absorption calculations, vacuum polarization and Coulomb effects are neglected. The wave function inside the core radius,  $r_0$ , is taken to be zero.<sup>109</sup>

### APPENDIX III

#### NUMERICAL SOLUTION OF THE TENSOR-COUPLED SCHRÖDINGER EQUATIONS

Nucleon-nucleon potentials containing tensor terms lead in general to coupled differential equations of the form,

$$\frac{d^2 u}{dr^2} + f(r)u + g(r)w = 0$$

and

$$\frac{d^2 w}{dr^2} + h(r)w + g(r)u = 0. \quad (\text{III-1})$$

Here  $u(r)$  and  $w(r)$  are wave functions of orbital angular momentum number,  $L = J \mp 1$ . In this appendix we will use abbreviations typified by

$$\begin{aligned} f &= f(r), \\ f_{\pm} &= f(r \pm \epsilon), \\ u^n &= \left. \frac{d^n u}{dr^n} \right|_r \end{aligned}$$

and

$$u_{\pm}^n = \left. \frac{d^n u}{dr^n} \right|_{r \pm \epsilon}. \quad (\text{III-2})$$

The quantities,  $u_+$  and  $u_-$ , may be expanded in Taylor series about  $r$  to give

$$u_{\pm} = u \pm \epsilon u' + \frac{\epsilon^2}{2!} u'' \pm \frac{\epsilon^3}{3!} u''' + \frac{\epsilon^4}{4!} u^{(4)} \pm \frac{\epsilon^5}{5!} u^{(5)} + \dots \quad (\text{III-3})$$

Adding the two expressions in Eq. (III-3) and dropping all terms of order six and higher gives an expression correct to fifth order in  $\epsilon$ . This can be written as

$$u_+ + u_- - 2u = \epsilon^2 u^2 + \frac{\epsilon^4}{12} u^4. \quad (\text{III-4})$$

An exact expression for the second derivative term,  $u^2$ , is given by Eq. (III-1). The fourth derivative term,  $u^4$ , may be obtained by a Taylor series expansion of the second derivative function,  $u^2$ , about  $r$ . This result is:

$$u^4 = \bar{\epsilon}^{-2} \{ u_+^2 + u_-^2 - 2u^2 \} - \frac{\epsilon^2}{12} u^6 - \dots. \quad (\text{III-5})$$

By dropping terms of order six and higher and making use of the definition for  $u^2$  in Eq. (III-1), an approximation for  $u^4$  can be written as

$$u^4 = \bar{\epsilon}^{-2} \{ -f_+ u_+ - g_+ \omega_+ - f_- u_- - g_- \omega_- + 2fu + 2f\omega \}. \quad (\text{III-6})$$

Substituting Eq. (III-6) and the first expression in Eq. (III-1) into Eq. (III-4) and rearranging gives the result,

$$u_+ \left( 1 + \frac{\epsilon^2}{12} f_+ \right) + u_- \left( 1 + \frac{\epsilon^2}{12} f_- \right) + u \left( -2 + \frac{5\epsilon^2}{6} f \right) = -\epsilon^2 \left\{ \frac{5}{6} g\omega + \frac{1}{12} (g_+ \omega_+ + g_- \omega_-) \right\}. \quad (\text{III-7})$$

From the symmetry of Eq. (III-1) a similar result is obtained in  $w$ :

$$\begin{aligned} \omega_+ \left(1 + \frac{\epsilon^2}{12} h_+\right) + \omega_- \left(1 + \frac{\epsilon^2}{12} h_-\right) + w \left(-2 + \frac{5\epsilon^2}{6} h\right) = \\ - \epsilon^2 \left\{ \frac{5}{6} g u + \frac{1}{12} (g_+ u_+ + g_- u_-) \right\}. \end{aligned} \quad (\text{III-8})$$

Now, assuming an outward integration, Eq. (III-8) is solved for  $w_+$ , and the result substituted into Eq. (III-7), which upon rearrangement becomes

$$\begin{aligned} u_+ \left\{ 1 + \frac{\epsilon^2}{12} f_+ - \frac{\epsilon^4}{144} \frac{g_+^2}{\left(1 + \frac{\epsilon^2}{12} h_+\right)} \right\} + u_- \left\{ 1 + \frac{\epsilon^2}{12} f_- - \frac{\epsilon^4}{144} \frac{g_+ g_-}{\left(1 + \frac{\epsilon^2}{12} h_+\right)} \right\} \\ + u \left\{ -2 + \frac{5\epsilon^2}{6} f - \frac{5\epsilon^4}{72} \frac{g_+ g}{\left(1 + \frac{\epsilon^2}{12} h_+\right)} \right\} = \\ \omega_+ \left\{ -\frac{5\epsilon^2}{6} g + \frac{\epsilon^2}{12} g_+ \frac{\left(-2 + \frac{5\epsilon^2}{6} h\right)}{\left(1 + \frac{\epsilon^2}{12} h_+\right)} \right\} + \omega_- \left\{ -\frac{\epsilon^2}{12} g_- + \frac{\epsilon^2}{12} g_+ \frac{\left(1 + \frac{\epsilon^2}{12} h_-\right)}{\left(1 + \frac{\epsilon^2}{12} h_+\right)} \right\}. \end{aligned} \quad (\text{III-9})$$

Useful abbreviations are now seen to be

$$H = 1 + \frac{\epsilon^2}{12} h ,$$

$$F = 1 + \frac{\epsilon^2}{12} f$$

and

$$G = \frac{\epsilon^2}{12} g . \quad (\text{III-10})$$

Equations (III-9) and (III-8) are rewritten using these abbreviations to give the final result,

$$u_+ = (F_+ H_+ - G_+ G_+)^{-1} \left\{ -(F_- H_+ - G_- G_+) u_- - (G_- H_+ - G_+ H_-) \omega_- \right. \\ \left. + \{ 12 H_+ - 10 (F H_+ - G_+ G) \} u - \{ 12 G_+ - 10 (G_+ H - G H_+) \} \omega \right\}$$

and

$$\omega_+ = -H_+^{-1} \left\{ H_- \omega_- + (10H - 12)\omega + G_- u_- + 10G u + G_+ u_+ \right\}. \quad (III-11)$$

The expressions in Eq. (III-11) are meant to be solved sequentially, first for  $u_+$  and then for  $\omega_+$ , and represent a fifth-order, three-point solution of Eq. (III-1). The functions,  $F$ ,  $G$  and  $H$ , are easily calculable given a potential function. The uncoupled scattering case is solved using the degenerate form of Eq. (III-11), as is the Klein-Gordon equation used for the pion optical model in Chapter 4. In the latter case, the function,  $f$ , and therefore the wave function,  $u$ , are complex.

## APPENDIX IV

### THE DEUTERON EIGENVALUE

Deuteron wave functions are obtained from phenomenological potentials by solving Schrödinger's equation while systematically varying the binding energy until the boundary conditions are satisfied. The outer boundary conditions for the S- and D-orbitals can be expressed outside the nuclear potential as

$$u(r) \sim e^{-kr}$$

and

$$w(r) \sim e^{-kr} \left( 1 + \frac{3}{kr} + \frac{3}{(kr)^2} \right). \quad (\text{IV-1})$$

The inner boundary conditions depend on the potential model chosen. The wave function vanishes at  $r_0$  in hard-core models such as the Hamada-Johnston, while the Boundary Condition Model derives its name from a matrix logarithmic-derivative representation.

Two different techniques were developed to search for energy eigenvalues. The first is limited to hard-core models. An initial guess is made for the eigenvalue, and two independent ratios of the boundary conditions expressed in Eq. (IV-1) are chosen. The differential equations are numerically integrated inward to obtain two linearly independent solutions,  $(u_1, w_1)$  and  $(u_2, w_2)$ . These wave functions are temporarily normalized. A function, defined by

$$\delta = \left\{ u_1(r_0) + a u_2(r_0) \right\}^2 + \left\{ w_1(r_0) + a w_2(r_0) \right\}^2, \quad (\text{IV-2})$$

is then calculated, with the coefficient,  $a$ , chosen to minimize the value of  $\delta$ . The approach is then to drive  $\delta$  to zero by a systematic variation of the binding energy, thus satisfying the inner boundary conditions. Following convergence to  $\delta < \epsilon'$ , the linear combination,

$$(u, \omega) = (u_1, \omega_1) + a(u_2, \omega_2), \quad (\text{IV-3})$$

is taken and normalized according to

$$\int_0^{\infty} dr (u^2(r) + \omega^2(r)) = 1. \quad (\text{IV-4})$$

The D-state fraction is then given by

$$P_D = \int_0^{\infty} dr \omega^2(r), \quad (\text{IV-5})$$

while an experimentally better-known quantity, the quadrupole moment, is obtained from

$$Q = \frac{\sqrt{1}}{\sqrt{50}} \int_0^{\infty} dr r^2 u(r) \omega(r) - \frac{1}{20} \int_0^{\infty} dr r^2 \omega^2(r). \quad (\text{IV-6})$$

A second and more general technique involves outward integration beginning with two independently chosen sets of inner boundary conditions.

At some radius beyond the nuclear potential, the two trial wave functions,  $(u_1, w_1)$  and  $(u_2, w_2)$ , can each be written as a linear combination of increasing ( $f_L^+$ ) and decreasing ( $f_L^-$ ) exponential functions. The total wave function is in turn made up of a linear combination of the two trial wave functions. We therefore have

$$(u, w) = c_1 (u_1, w_1) + c_2 (u_2, w_2),$$

$$(u, w) = c_1 (A_{11} f_0^- + B_{11} f_0^+, A_{21} f_2^- + B_{21} f_2^+) + c_2 (A_{12} f_0^- + B_{12} f_0^+, A_{22} f_2^- + B_{22} f_2^+),$$

or, rewriting this last expression,

$$(u, w) = \left( (c_1 A_{11} + c_2 A_{12}) f_0^-, (c_1 A_{21} + c_2 A_{22}) f_2^- \right) + \left( (c_1 B_{11} + c_2 B_{12}) f_0^+, (c_1 B_{21} + c_2 B_{22}) f_2^+ \right). \quad (\text{IV-7})$$

The boundary conditions expressed in Eq. (IV-1) will be satisfied if the increasing exponential components vanish. This requires that

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0,$$

or equivalently, that

$$\begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = 0.$$

(IV-8)



Thus the binding energy value is systematically varied to drive the determinant of the B-matrix to zero. The A- and B-matrices are found by matching the asymptotic forms to the trial wave functions at  $r$  and  $r - \epsilon$ . Equations (IV-4) through (IV-6) are again applicable.

## APPENDIX V

### THE FINAL-STATE WAVE FUNCTION

In this appendix the final state wave function,  $\psi_f^-$ , for the reaction  $\pi^+ + d \rightarrow p + p$  is obtained. First the nucleon-nucleon scattering process is considered, then symmetrization is used to take into account the identity of the two protons, and finally the desired result is obtained by a process equivalent to time reversal.

The basis used in the solution of the Schrödinger equation is the spin-angle function,  $y_{JLS}^M$ . The expansion of the wave function is done in helicity states; that is, angular momentum projections are measured in the direction of the outgoing momentum vector,  $\vec{k}$ . The region beyond the centrifugal barrier is denoted by the subscript,  $r \rightarrow \infty$ . The region where the centrifugal term is not negligible but the nuclear potential has essentially vanished is designated by  $V(r) \sim 0$ . Finally, absence of a subscript indicates a function correct throughout the space.

The scattering wave function for two spin one-half particles in the absence of interaction may be written as a product of plane wave and spin function, according to

$$\psi = e^{i\vec{k}\cdot\vec{r}} \chi_s^M, \quad (V-1)$$

where the spin function is defined in terms of its components by

$$\chi_s^M = \sum_{M_1, M_2} \begin{bmatrix} S & M \\ S_1 & M_1 \\ S_2 & M_2 \end{bmatrix} |S_1, M_1, S_2, M_2\rangle. \quad (V-2)$$

In this case we have  $S_1 = S_2 = 1/2$ . The spin-angle function is defined by the relationship,

$$Y_{JLS}^M = \sum_{M_L, M_S} \begin{bmatrix} J & M \\ L & M_L \\ S & M_S \end{bmatrix} Y_L^{M_L} \chi_S^{M_S}. \quad (V-3)$$

By expanding the plane wave into partial waves and making use of the relation,

$$Y_L^0 \chi_S^{M_S} = \sum_{M, J=|L-S|}^{L+S} \begin{bmatrix} J & M \\ L & 0 \\ S & M_S \end{bmatrix} Y_{JLS}^M, \quad (V-4)$$

we obtain the expression,

$$e^{i\vec{k}\cdot\vec{r}} \chi_S^{M_S} = \sqrt{\pi} \sum_{M, J, L} (2L+1)^{1/2} i^L \{h_L^2(kr) + h_L^1(kr)\} \begin{bmatrix} J & M \\ L & 0 \\ S & M_S \end{bmatrix} Y_{JLS}^M. \quad (V-5)$$

The spherical Hankel functions are given in terms of spherical Bessel and Neumann functions by

$$h_L^1(kr) = h_L^{*2}(kr) = j_L(kr) + i\eta_L(kr), \quad (V-6)$$

or, in the limit,  $r \rightarrow \infty$ , by

$$h'_l(kr) = h_l^{*2}(kr) \xrightarrow{r \rightarrow \infty} -\frac{i}{kr} e^{i(kr - \pi L/2)}. \quad (V-7)$$

In this case Eq. (V-5) can be written as

$$e^{i\vec{k}\cdot\vec{r}} \chi_s^{M_s} = \frac{\sqrt{\pi}}{kr} \sum_{M, J, L} (2L+1)^{1/2} i^{L+1} \left\{ e^{-i(kr - \pi L/2)} - e^{+i(kr - \pi L/2)} \right\} \begin{bmatrix} J & M \\ L & 0 \\ S & M_s \end{bmatrix} Y_{JLS}^M. \quad (V-8)$$

Therefore the spherical Hankel function,  $h_L^2 (h_L^1)$ , may be identified with an incoming (outgoing) spherical wave. In the absence of tensor coupling and for a real Hamiltonian, an asymptotic form for the wave function can be obtained by inserting a unitary factor to multiply the outgoing spherical wave. With tensor coupling, however, orbital angular momentum is not in general conserved, and an incident wave of pure angular momentum,  $L$ , may result in several outgoing waves,  $L'$ . Thus a correct expression for the asymptotic wave function must be

$$\psi \xrightarrow{r \rightarrow \infty} \frac{\sqrt{\pi}}{kr} \sum_{MJL L'} (2L+1)^{1/2} i^{L+1} \begin{bmatrix} J & M \\ L & 0 \\ S & M_s \end{bmatrix} \left\{ e^{-i(kr - \pi L/2)} Y_{JLS}^M - S_{JLL's} e^{+i(kr - \pi L'/2)} Y_{JL's}^M \right\}. \quad (V-9)$$

The total angular momentum,  $J$ , and component,  $M$ , are conserved in the absence of external interactions. The spin quantum number,  $S$ , is assumed conserved. For the triplet ( $S = 1$ ) case, possible values of  $L'$  include  $J - 1$ ,  $J$  and  $J + 1$ . If the nuclear force conserves parity, however, only the  $J - 1$  and  $J + 1$  orbitals may be coupled. Therefore, both the triplet with  $L = J$  and the singlet ( $S = 0$ ) states are uncoupled (the tensor operator even vanishes on the singlet state), and a component of either can be written simply as

$$\psi_{JMLS} = \lim_{r \rightarrow \infty} \frac{\sqrt{\pi}}{kr} (2L+1)^{1/2} i^{L+1} \left\{ e^{-i(kr - \pi L/2)} - S_L e^{+i(kr - \pi L/2)} \right\} \begin{bmatrix} J & M \\ L & 0 \\ S & M \end{bmatrix} Y_{JLS}^M \quad (V-10)$$

The unitary factor,  $S_L$ , is written as

$$S_L = e^{2i\delta_L} \quad (V-11)$$

where  $\delta_L$  is the phase shift. We desire the wave function,  $u_L$ , to be real and normalized according to the asymptotic relationship,

$$u_L(r) = \lim_{r \rightarrow \infty} \sin(kr - \pi L/2 + \delta_L) \quad (V-12)$$

Then the uncoupled wave function correct over all space may be written as

$$\psi_{JMLS} = \frac{2\sqrt{\pi}}{k} (2L+1)^{1/2} i^L e^{i\delta_L} \begin{bmatrix} J & M \\ L & 0 \\ S & M \end{bmatrix} \frac{u_L(r)}{r} Y_{JLS}^M \quad (V-13)$$

In practice, a numerical function,  $\tilde{u}_L$ , is calculated by integrating Schrödinger's equation using the method presented in Appendix III. The desired wave function,  $u_L$ , is obtained by matching the calculated function to the asymptotic form. For increased accuracy, this matching is done within the centrifugal barrier. In this region, the asymptotic form analogous to Eq. (V-12) is

$$u_L(r) = c \tilde{u}_L(r) = kr \left\{ \cos \delta_L j_L(kr) - \sin \delta_L \eta_L(kr) \right\} \quad (V-14)$$

The phase shift is then determined from the relationship,

$$\tan \delta_L = \frac{(r-\epsilon)\tilde{u}_L(r)\dot{j}_L(k(r-\epsilon)) - r\tilde{u}_L(r-\epsilon)\dot{j}_L(kr)}{(r-\epsilon)\tilde{u}_L(r)\eta_L(k(r-\epsilon)) - r\tilde{u}_L(r-\epsilon)\eta_L(kr)} \quad (V-15)$$

With the phase shift known, the normalization constant relating  $u_L$  and  $\tilde{u}_L$  is obtained directly from Eq. (V-14).

The remaining triplet  $L = J \pm 1$  case involves solution of coupled differential equations. The asymptotic form is contained within Eq. (V-9) and can be written, for a particular value of  $J$ , as

$$\psi_{JML=J\mp 11} = \left\{ a' e^{-i(kr-\pi(J-1)/2)} - (s_{11}a' + s_{21}b') e^{+i(kr-\pi(J-1)/2)} \right\} Y_{JJ-1}^M + \left\{ b' e^{-i(kr-\pi(J+1)/2)} - (s_{12}a' + s_{22}b') e^{+i(kr-\pi(J+1)/2)} \right\} Y_{JJ+1}^M,$$

where

$$a' = \frac{\sqrt{\pi}}{kr} (2J-1)^{1/2} i^J \begin{bmatrix} J & M \\ J-1 & 0 \\ 1 & M \end{bmatrix}$$

and

$$b' = \frac{\sqrt{\pi}}{kr} (2J+3)^{1/2} i^{J+2} \begin{bmatrix} J & M \\ J+1 & 0 \\ 1 & M \end{bmatrix} . \quad (V-16)$$

This asymptotic form could be used directly for matching and normalization; however, the resulting function would be complex, and what is even more objectionable, dependent upon the projection,  $M$ , through the Clebsch-Gordan coefficients.

A more useful parameterization is that of Blatt and Biedenharn,<sup>110</sup> who define eigenstates of scattering for which the ratio of outgoing spherical waves equals the ratio of incoming spherical waves. Such states are characterized by the absence of mixing of partial waves, and hence by an equivalent phase shift for each. The two partial waves associated with  $L = J \mp 1$  in Eq. (V-16) may be written, respectively, as

$$u \underset{r \rightarrow \infty}{=} A_1 e^{-i(kr - \pi(J-1)/2)} - B_1 e^{+i(kr - \pi(J-1)/2)}$$

and

$$w \underset{r \rightarrow \infty}{=} A_2 e^{-i(kr - \pi(J+1)/2)} - B_2 e^{+i(kr - \pi(J+1)/2)} ,$$

where

$$B_1 = S_{11}A_1 + S_{12}A_2$$

and

$$B_2 = S_{21}A_1 + S_{22}A_2 \quad (V-17)$$

The S-matrix must be symmetric and unitary, and can be expressed as

$$S = \begin{pmatrix} \cos^2 \epsilon_J e^{2i\alpha} + \sin^2 \epsilon_J e^{2i\beta} & \cos \epsilon_J \sin \epsilon_J (e^{2i\alpha} - e^{2i\beta}) \\ \cos \epsilon_J \sin \epsilon_J (e^{2i\alpha} - e^{2i\beta}) & \sin^2 \epsilon_J e^{2i\alpha} + \cos^2 \epsilon_J e^{2i\beta} \end{pmatrix} \quad (V-18)$$

In this expression,  $\alpha$  and  $\beta$  are the eigenstate phase shifts, and  $\epsilon_J$  is commonly referred to as the mixing parameter. The previously given definition of an eigenstate evidently requires either that

$$A_2 = A_1 \sin \epsilon_J / \cos \epsilon_J$$

or that

$$A_2 = -A_1 \cos \epsilon_J / \sin \epsilon_J,$$

yielding, respectively,

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = e^{2i\alpha} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

or

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = e^{2i\beta} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad (V-19)$$



Making use of Eqs. (V-17), (V-18) and (V-19), the eigenstate wave functions are defined to be

$$\begin{aligned} u_\alpha &= \cos \epsilon_J \sin(kr - \pi(J-1)/2 + \alpha), \\ \omega_\alpha &= \sin \epsilon_J \sin(kr - \pi(J+1)/2 + \alpha), \\ u_\beta &= -\sin \epsilon_J \sin(kr - \pi(J-1)/2 + \beta) \end{aligned}$$

and

$$\omega_\beta = \cos \epsilon_J \sin(kr - \pi(J+1)/2 + \beta) \quad (V-20)$$

A linear combination of these eigenstate wave functions is now required to satisfy Eq. (V-17). First we consider an incident spherical wave of  $L = J - 1$ . From Eqs. (V-17) and (V-18), the resulting wave functions must be

$$u = e^{-i(kr - \pi(J-1)/2)} - (\cos^2 \epsilon_J e^{2i\alpha} + \sin^2 \epsilon_J e^{2i\beta}) e^{+i(kr - \pi(J-1)/2)}$$

and

$$\omega = 0 - \sin \epsilon_J \cos \epsilon_J (e^{2i\alpha} - e^{2i\beta}) e^{+i(kr - \pi(J+1)/2)} \quad (V-21)$$

Similarly, for incident wave of  $L = J + 1$ , the wave functions are

$$u = 0 - \sin \epsilon_J \cos \epsilon_J (e^{2i\alpha} - e^{2i\beta}) e^{+i(kr - \pi(J-1)/2)}$$

and

$$\omega = e^{-i(kr - \pi(J+1)/2)} - (\sin^2 \epsilon_J e^{2i\alpha} + \cos^2 \epsilon_J e^{2i\beta}) e^{+i(kr - \pi(J+1)/2)}$$

(V-22)

These are then the required results, once normalization is provided by Eq. (V-16). The wave function is a linear combination of Eqs. (V-21) and (V-22); in terms of the eigenstate results of Eq. (V-20) it is given by

$$\begin{aligned} \psi_{JM J \neq 11} = & \\ & \left\{ -2i(a' \cos \epsilon_J + b' \sin \epsilon_J) e^{i\alpha} u_\alpha + 2i(a' \sin \epsilon_J - b' \cos \epsilon_J) e^{i\beta} u_\beta \right\} Y_{JJ-11}^M \\ & + \left\{ -2i(a' \cos \epsilon_J + b' \sin \epsilon_J) e^{i\alpha} w_\alpha + 2i(a' \sin \epsilon_J - b' \cos \epsilon_J) e^{i\beta} w_\beta \right\} Y_{JJ+11}^M . \end{aligned} \quad \text{V-23}$$

There remains the detail of the calculation of  $\alpha$ ,  $\beta$ ,  $\epsilon_J$  and the eigenstate wave functions. In the presence of the centrifugal barrier, the  $\alpha$  eigenstate functions can be written as

$$u_\alpha = \underset{V(r) \sim 0}{kr \cos \epsilon_J} \left\{ \cos \alpha j_{J-1}(kr) - \sin \alpha \eta_{J-1}(kr) \right\}$$

and

$$w_\alpha = \underset{V(r) \sim 0}{kr \sin \epsilon_J} \left\{ \cos \alpha j_{J+1}(kr) - \sin \alpha \eta_{J+1}(kr) \right\} . \quad \text{V-24}$$

Using two different sets of boundary conditions, two linearly independent functions,  $(\tilde{u}_1, \tilde{w}_1)$  and  $(\tilde{u}_2, \tilde{w}_2)$ , are obtained by integration of the Schrödinger equation. A linear combination of the two solutions is taken and matched to  $(u_\alpha, w_\alpha)$  at an outer radius,  $r$ , and  $r - \epsilon$ . The four resulting equations which contain four unknowns are

$$c_1 \tilde{u}_1(r) + c_2 \tilde{u}_2(r) = \underset{V(r) \sim 0}{kr \cos \epsilon_J} \left\{ \cos \alpha j_{J-1}(kr) - \sin \alpha \eta_{J-1}(kr) \right\} ,$$

$$c_1 \tilde{w}_1(r) + c_2 \tilde{w}_2(r) = kr \sin \epsilon_J \left\{ \cos \alpha j_{J+1}(kr) - \sin \alpha \eta_{J+1}(kr) \right\},$$

$$c_1 \tilde{u}_1(r-\epsilon) + c_2 \tilde{u}_2(r-\epsilon) = \\ k(r-\epsilon) \cos \epsilon_J \left\{ \cos \alpha j_{J-1}(k(r-\epsilon)) - \sin \alpha \eta_{J-1}(k(r-\epsilon)) \right\}$$

and

$$c_1 \tilde{w}_1(r-\epsilon) + c_2 \tilde{w}_2(r-\epsilon) = \\ k(r-\epsilon) \sin \epsilon_J \left\{ \cos \alpha j_{J+1}(k(r-\epsilon)) - \sin \alpha \eta_{J+1}(k(r-\epsilon)) \right\} \quad (A-25)$$

A solution to these equations is found by elimination of  $c_1$  and  $c_2$ , which results in two expressions involving the tangents of  $\alpha$  and  $\epsilon_J$ .

These expressions are

$$\tan \epsilon_J = \frac{a_1 + a_2 \tan \alpha}{a_3 + a_4 \tan \alpha}$$

and

$$\tan \epsilon_J = \frac{a_5 + a_6 \tan \alpha}{a_7 + a_8 \tan \alpha},$$

where

$$a_1 = (r-\epsilon) j_{J-1}(k(r-\epsilon)) \left\{ \tilde{u}_1(r) \tilde{w}_2(r) - \tilde{u}_2(r) \tilde{w}_1(r) \right\} \\ - r j_{J-1}(kr) \left\{ \tilde{u}_1(r-\epsilon) \tilde{w}_2(r) - \tilde{u}_2(r-\epsilon) \tilde{w}_1(r) \right\},$$

$$a_2 = -(r-\epsilon) \eta_{J-1}(k(r-\epsilon)) \left\{ \tilde{u}_1(r) \tilde{w}_2(r) - \tilde{u}_2(r) \tilde{w}_1(r) \right\} \\ + r \eta_{J-1}(kr) \left\{ \tilde{u}_1(r-\epsilon) \tilde{w}_2(r) - \tilde{u}_2(r-\epsilon) \tilde{w}_1(r) \right\},$$

$$a_3 = r \underset{J+1}{f}(kr) \left\{ \tilde{u}_1(r) \tilde{u}_2(r-\epsilon) - \tilde{u}_2(r) \tilde{u}_1(r-\epsilon) \right\} ,$$

$$a_4 = -r \underset{J+1}{\eta}(kr) \left\{ \tilde{u}_1(r) \tilde{u}_2(r-\epsilon) - \tilde{u}_2(r) \tilde{u}_1(r-\epsilon) \right\} ,$$

$$a_5 = r \underset{J-1}{f}(kr) \left\{ \tilde{w}_1(r) \tilde{w}_2(r-\epsilon) - \tilde{w}_2(r) \tilde{w}_1(r-\epsilon) \right\} ,$$

$$a_6 = -r \underset{J-1}{\eta}(kr) \left\{ \tilde{w}_1(r) \tilde{w}_2(r-\epsilon) - \tilde{w}_2(r) \tilde{w}_1(r-\epsilon) \right\} ,$$

$$a_7 = (r-\epsilon) \underset{J+1}{f}(k(r-\epsilon)) \left\{ \tilde{w}_1(r) \tilde{u}_2(r) - \tilde{w}_2(r) \tilde{u}_1(r) \right\} \\ - r \underset{J+1}{f}(kr) \left\{ \tilde{w}_1(r-\epsilon) \tilde{u}_2(r) - \tilde{u}_1(r) \tilde{w}_2(r-\epsilon) \right\}$$

and

$$a_8 = -(r-\epsilon) \underset{J+1}{\eta}(k(r-\epsilon)) \left\{ \tilde{w}_1(r) \tilde{u}_2(r) - \tilde{w}_2(r) \tilde{u}_1(r) \right\} \\ + r \underset{J+1}{\eta}(kr) \left\{ \tilde{w}_1(r-\epsilon) \tilde{u}_2(r) - \tilde{u}_1(r) \tilde{w}_2(r-\epsilon) \right\} . \quad (V-26)$$

Equating the two expressions for  $\epsilon_J$  yields a quadratic in the tangent of  $\alpha$ , which is given by

$$a_\alpha \tan^2 \alpha + b_\alpha \tan \alpha + c_\alpha = 0 ,$$

where

$$a_\alpha = a_2 a_8 - a_4 a_6 ,$$

$$b_\alpha = a_1 a_8 + a_2 a_7 - a_3 a_6 - a_4 a_5$$

and

$$c_\alpha = a_1 a_7 - a_3 a_5 . \quad (V-27)$$

The apparent ambiguity of this equation is resolved when it is remembered that the mixing parameter normalization of the  $\alpha$  and  $\beta$  eigenstates is arbitrary; that is, we could have written the  $\beta$  eigenstate using  $\cos \epsilon'_J$  and  $\sin \epsilon'_J$  instead of  $-\sin \epsilon_J$  and  $\cos \epsilon_J$ , where  $\epsilon'_J = \epsilon_J - \pi/2$ . Therefore the roots of Eq. (V-27) give both  $\alpha$  and  $\beta$ .

The usefulness of the eigenstate parameterization for numerical work is evident; however, experimental values are usually quoted using one of several other systems. An identification is easily made with the U-matrix formalism<sup>111</sup> through the relations,

$$\begin{aligned}\theta_- + \theta_+ &= \alpha + \beta, \\ \tan(\theta_- - \theta_+) &= \cos 2\epsilon_J \tan(\alpha - \beta)\end{aligned}$$

and

$$\rho = \sin 2\epsilon_J \sin(\alpha - \beta). \quad (\text{V-28})$$

Similarly, the nuclear bar parameters<sup>112</sup> are obtained in turn from the relations,

$$\begin{aligned}\bar{\delta}_{J-1} &= \theta_-, \\ \bar{\delta}_{J+1} &= \theta_+\end{aligned}$$

and

$$\sin 2\bar{\epsilon} = \rho. \quad (\text{V-29})$$

It is yet necessary to symmetrize the wave functions, taking into account that the spin one-half particles in the final state are identical,

and thus dynamically equivalent. The symmetrized wave function is obtained from the unsymmetrized function by taking<sup>113</sup>

$$\psi_s = \sqrt{2} \mathcal{S} \psi, \quad (\text{V-30})$$

where the symmetrization operator is

$$\mathcal{S} = \frac{1}{2} \{ 1 + \delta_{Q_{12}} Q_{12} \}. \quad (\text{V-31})$$

The permutation operator is easily shown to give

$$Q_{12} \psi(\vec{r}) \chi_s^{M_s} = (-1)^{l-s} \psi(-\vec{r}) \chi_s^{M_s}, \quad (\text{V-32})$$

while for Fermi-Dirac particles,  $\delta_{Q_{12}} = -1$ .

Finally, the wave function must be obtained for the inverse of the scattering process. This may be obtained formally through the expression,<sup>114</sup>

$$\psi_{\vec{k} s M_s}^- = T \psi_{-\vec{k} s - M_s}^-, \quad (\text{V-33})$$

where  $T$  is the time reversal operator. The same result may be obtained in a more straightforward manner by considering Eq. (V-10) and the inverse process. In this case, it is the incoming spherical wave which must be modified by a phase change. Equation (V-10) now should be written as

$$\psi_{JMLS}^- = \frac{\sqrt{\pi}}{kr} (2L+1)^{1/2} i^{L+1} \left\{ S_L e^{-i(kr-\pi L/2)} - e^{+i(kr-\pi L/2)} \right\} \begin{bmatrix} J & M \\ L & 0 \\ S & M \end{bmatrix} Y_{JLS}^M . \quad (V-34)$$

The Schrödinger equation which we solve, however, is time-independent. To recover the form of Eq. (V-12), we simply let

$$S_L \rightarrow e^{-2i\delta_L} . \quad (V-35)$$

The result for the singlet and triplet uncoupled cases is

$$\psi_{JMLS}^- = \frac{2\sqrt{\pi}}{k} (2L+1)^{1/2} i^L e^{-i\delta_L} \begin{bmatrix} J & M \\ L & 0 \\ S & M \end{bmatrix} \frac{u_L(r)}{r} Y_{JLS}^M . \quad (V-36)$$

A similar result is found for the coupled triplet case.

With these results, and those required for symmetrization, the wave function used for the final state becomes, for the singlet and uncoupled triplet case,

$$\psi_{JMLS}^- = \frac{2\sqrt{2\pi}}{k} (2L+1)^{1/2} i^L e^{-i\delta_L} \begin{bmatrix} J & M \\ L & 0 \\ S & M \end{bmatrix} \frac{u_L(r)}{r} Y_{JLS}^M , \quad (V-37)$$

and for the coupled triplet case,

$$\psi_{JM, J \neq 1}^- = \left\{ (a \cos \epsilon_J + b \sin \epsilon_J) e^{-i\alpha \frac{u_\alpha(r)}{r}} - (a \sin \epsilon_J - b \cos \epsilon_J) e^{-i\beta \frac{u_\beta(r)}{r}} \right\} Y_{J, J-1}^M + \left\{ (a \cos \epsilon_J + b \sin \epsilon_J) e^{-i\alpha \frac{u_\alpha(r)}{r}} - (a \sin \epsilon_J - b \cos \epsilon_J) e^{-i\beta \frac{u_\beta(r)}{r}} \right\} Y_{J, J+1}^M,$$

where

$$a = \frac{2\sqrt{2\pi}}{k} (2J-1)^{1/2} i^{J-1} \begin{bmatrix} J & M \\ J-1 & 0 \\ 1 & M \end{bmatrix}$$

and

$$b = \frac{2\sqrt{2\pi}}{k} (2J+3)^{1/2} i^{J+1} \begin{bmatrix} J & M \\ J+1 & 0 \\ 1 & M \end{bmatrix} \quad (V-38)$$



## APPENDIX VI

### EVALUATION OF THE MATRIX ELEMENT

In this appendix the matrix elements for the nuclear transition operator defined in Eq. (2-7) are given. The evaluation is carried out in detail for a typical term, in particular, the p-wave scattering, s-wave absorption portion of  $T^4$ . Results are then simply stated for the remaining angular integrals. These angular integrals,  $R_i$ , are common to the formalisms developed in Chapters 2, 3 and 4. Differences in the formalisms are limited to the radial integrals,  $I_i$ , which are therefore stated explicitly in the appropriate chapters.

Before the evaluation of this term in the matrix element, some general results and definitions are stated. The isospin variables in the transition operator,  $T$ , have been left as single-nucleon operators. Therefore the isospin contribution to the matrix element is obtained by expanding the initial and final isospin states in the bracket, according to

$$\langle T' M_T' | \mathcal{T}_{op} | T M_T \rangle = \sum_{\substack{\mu_1' \mu_2' \\ \mu_1 \mu_2}} \begin{bmatrix} T' M_T' \\ \mathcal{T}_1' \mu_1' \\ \mathcal{T}_2' \mu_2' \end{bmatrix} \begin{bmatrix} T M_T \\ \mathcal{T}_1 \mu_1 \\ \mathcal{T}_2 \mu_2 \end{bmatrix} \langle \mathcal{T}_1' \mu_1' \mathcal{T}_2' \mu_2' | \mathcal{T}_{op} | \mathcal{T}_1 \mu_1 \mathcal{T}_2 \mu_2 \rangle. \quad (\text{VI-1})$$

The choice of overall signs for the pion field components given in Eq. (I-1) forces use of the definition,

$$\tau^{\pm} = \pm \frac{1}{\sqrt{2}} (\tau^1 \pm i\tau^2). \quad (\text{VI-2})$$

The components,  $\tau^i$ , are the Pauli matrices. The isospin matrix elements are then easily evaluated using Eq. (VI-1). The particular results required are

$$\langle 1, 1 | \tau_1^+ | 0, 0 \rangle = -1 ,$$

$$\langle 1, 1 | \tau_2^+ | 0, 0 \rangle = +1 ,$$

$$\langle 1, 1 | \tau_1^+ \tau_2^3 | 0, 0 \rangle = -1$$

and

$$\langle 1, 1 | \tau_1^3 \tau_2^+ | 0, 0 \rangle = +1 . \quad (\text{VI-3})$$

All other vector operators are written as spherical tensors, which are understood to be of rank one. The subscript is then free to be used, wherever convenient, as a nucleon label, while the superscript denotes the tensor component. The usual spherical tensor definitions are

$$A^{\pm 1} = \mp \frac{1}{\sqrt{2}} (A^x \pm iA^y)$$

and

$$A^0 = A^z. \quad (\text{VI-4})$$

These components can also be written in terms of the spherical harmonics:

$$A^\alpha = A \sqrt{\frac{4\pi}{3}} Y_1^\alpha(\Omega). \quad (\text{VI-5})$$

By analogy with Eq. (VI-4), components of the gradient operator are given by

$$\nabla^{\pm 1} = \mp \frac{1}{\sqrt{2}} (\nabla^x \pm i \nabla^y)$$

and

$$\nabla^0 = \nabla^z. \quad (\text{VI-6})$$

However, only the z-component will be required due to use of the Wigner-Eckart theorem.

The dot product of two vectors can be written using tensor components as

$$\bar{A} \cdot \bar{B} = \sum_{\alpha} (-1)^{\alpha} A^{\alpha} B^{-\alpha}. \quad (\text{VI-7})$$

To describe the dot-cross product, we define a tensor,  $\epsilon_{\alpha\beta\gamma}$ , in terms of the permutation indices.<sup>115</sup> The definition is

$$\epsilon_{1-10} = \epsilon_{01-1} = \epsilon_{-101} = +1,$$

$$\epsilon_{10-1} = \epsilon_{-110} = \epsilon_{0-11} = -1$$

and

$$\epsilon_{\alpha\beta\gamma} = 0, \quad (\text{VI-8})$$

otherwise. We note that this tensor can also be written as a Clebsch-Gordan coefficient:

$$\epsilon_{\alpha\beta\gamma} = (-1)^\alpha \sqrt{2} \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \\ 1 & \gamma \end{bmatrix}. \quad (\text{VI-9})$$

By making use of the definitions in Eqs. (VI-4) and (VI-8), it is easy to show that the vector dot-cross product is given by

$$\bar{A} \cdot \bar{B} \times \bar{C} = -i \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} A^\alpha B^\beta C^\gamma. \quad (\text{VI-10})$$

It is convenient to define two new symbols to describe certain spin matrix elements. The first is

$$S_\alpha = \langle S' M'_s | \sigma_1^\alpha + (-1)^\lambda \sigma_2^\alpha | S M_s \rangle. \quad (\text{VI-11})$$

Equation (VI-11) can be evaluated by defining spin sum and difference operators and using the Wigner-Eckart theorem directly. As an alternative, the spin states are expanded into products of single-nucleon states, and the Wigner-Eckart theorem is applied to each. This result is

$$S_{\alpha} = \sqrt{3} \sum_{\substack{M'_1, M'_2 \\ M_1, M_2}} \begin{bmatrix} S' M'_3 \\ S'_1 M'_1 \\ S'_2 M'_2 \end{bmatrix} \begin{bmatrix} S M_3 \\ S_1 M_1 \\ S_2 M_2 \end{bmatrix} \left\{ \delta_{M'_2, M_2} \begin{bmatrix} S'_1 M'_1 \\ S_1 M_1 \\ 1 \ \alpha \end{bmatrix} + (-1)^{\lambda} \begin{bmatrix} S'_2 M'_2 \\ S_2 M_2 \\ 1 \ \alpha \end{bmatrix} \delta_{M'_1, M_1} \right\}. \quad (\text{VI-12})$$

The second matrix element of interest is defined by

$$F_{\alpha\beta} = \langle S' M'_3 | \sigma_1^{\alpha} \sigma_2^{\beta} + (-1)^{\lambda} \sigma_1^{\beta} \sigma_2^{\alpha} | S M_3 \rangle. \quad (\text{VI-13})$$

Evaluation of this term is accomplished by means similar to those used to obtain Eq. (VI-12). The result is

$$F_{\alpha\beta} = 3 \sum_{\substack{M'_1, M'_2 \\ M_1, M_2}} \begin{bmatrix} S' M'_3 \\ S'_1 M'_1 \\ S'_2 M'_2 \end{bmatrix} \begin{bmatrix} S M_3 \\ S_1 M_1 \\ S_2 M_2 \end{bmatrix} \left\{ \begin{bmatrix} S'_1 M'_1 \\ S_1 M_1 \\ 1 \ \alpha \end{bmatrix} \begin{bmatrix} S'_2 M'_2 \\ S_2 M_2 \\ 1 \ \beta \end{bmatrix} + (-1)^{\lambda} \begin{bmatrix} S'_1 M'_1 \\ S_1 M_1 \\ 1 \ \beta \end{bmatrix} \begin{bmatrix} S'_2 M'_2 \\ S_2 M_2 \\ 1 \ \alpha \end{bmatrix} \right\}. \quad (\text{VI-14})$$

To illustrate the evaluation of the transition operator matrix element, a detailed calculation is given for the component involving p-wave scattering through  $H^4$  and s-wave absorption through the Galilean-invariant term of  $H^0$ . This nuclear transition operator is given by part of the last expression in Eq. (2-7) and will be designated  $T^4$ . It can be written as

$$\begin{aligned}
T^{4'} &= -\lambda_4 (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\bar{q}\cdot\bar{R}} \\
& (\tau_1^+ \tau_2^3 - \tau_1^3 \tau_2^+) \int \frac{d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}'\cdot\bar{r}}}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{q_0^2}{4} - q_0'^2\right)} \\
& \left(-\frac{q_0}{M}\right) \left\{ e^{-i\frac{\bar{q}'\cdot\bar{r}}{2}} \bar{\sigma}_1 \cdot \bar{k}' i \bar{\sigma}_2 \cdot \bar{q}' \times \bar{q} - e^{+i\frac{\bar{q}'\cdot\bar{r}}{2}} \bar{\sigma}_2 \cdot \bar{k}' i \bar{\sigma}_1 \cdot \bar{q}' \times \bar{q} \right\}. \quad (\text{VI-15})
\end{aligned}$$

With use of Eq. (VI-10), the vector product of nucleon spin and pion momenta in Eq. (VI-15) can be written in tensor components as

$$i \bar{\sigma} \cdot \bar{q}' \times \bar{q} = \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \sigma^\alpha q'^\beta q^\gamma. \quad (\text{VI-16})$$

The nucleon spin-momentum dot product can be expressed, using Eq. (VI-7), as

$$\bar{\sigma} \cdot \bar{k}' = \sum_{\alpha} (-1)^\alpha \sigma^\alpha k'^\alpha, \quad (\text{VI-17})$$

so that Eq. (VI-15) involves terms of the form,

$$\bar{\sigma}_1 \cdot \bar{k}' i \bar{\sigma}_2 \cdot \bar{q}' \times \bar{q} = \sum_{\alpha\beta\gamma} (-1)^\alpha \epsilon_{\alpha\beta\gamma} \bar{\sigma}_1 \cdot \bar{k}'^\alpha \bar{\sigma}_2 \cdot \bar{q}'^\beta q^\gamma. \quad (\text{VI-18})$$

Use is then made of Eq. (VI-5) to write the tensor component of the intermediate pion momentum,  $\vec{q}'$ , as a spherical harmonic. The angle is to be measured from the outgoing proton vector,  $\hat{k}'$ , so that

$$q'^{\beta} = q' \sqrt{\frac{4\pi}{3}} Y_{1}^{\beta}(\hat{q}', \hat{k}'). \quad (\text{VI-19})$$

With these results, Eq. (VI-15) becomes

$$\begin{aligned} T^{\lambda} &= -\lambda_{\pm} (4\pi)^{3/2} f \mu^{-4} \sqrt{\frac{1}{2b_0}} \frac{1}{\sqrt{V}} e^{i\vec{q}' \cdot \vec{R}} (\tau_1^{\pm} \tau_2^{\pm} - \tau_1^{\mp} \tau_2^{\mp}) \\ &\sum_{\alpha\beta\gamma\delta} (-1)^{\alpha} \epsilon_{\alpha\beta\gamma\delta} \left( e^{-i\frac{\vec{q}' \cdot \vec{r}}{2}} \sigma_1^{\alpha} \sigma_2^{\beta} - e^{+i\frac{\vec{q}' \cdot \vec{r}}{2}} \sigma_1^{\alpha} \sigma_2^{\beta} \right) q'^{\delta} \\ &\left( -\frac{q_0}{M} \right) k' \sqrt{\frac{4\pi}{3}} \int \frac{d\vec{q}' (\alpha^2 + q'^2) e^{i\vec{q}' \cdot \vec{r}} q' Y_{1}^{\beta}(\hat{q}', \hat{k}')}{(2\pi)^3 (\alpha^2 + q'^2) \left( \frac{q_0^2}{4} - q_0'^2 \right)}. \end{aligned} \quad (\text{VI-20})$$

We shall now evaluate the integral in Eq. (VI-20). The intermediate-pion wave function is expanded into partial waves, so that the integral becomes

$$\begin{aligned} &\int \frac{d\vec{q}' (\alpha^2 + q'^2) e^{i\vec{q}' \cdot \vec{r}} q' Y_{1}^{\beta}(\hat{q}', \hat{k}')}{(2\pi)^3 (\alpha^2 + q'^2) \left( \frac{q_0^2}{4} - q_0'^2 \right)} = \\ &\int \frac{d\vec{q}' (\alpha^2 + q'^2) 4\pi \sum_{\mu' \lambda'} i^{\lambda'} Y_{\lambda'}^{\mu'}(\hat{q}', \hat{k}') Y_{\lambda'}^{\mu'}(\hat{r}, \hat{k}') j_{\lambda'}(q' r) q' Y_{1}^{\beta}(\hat{q}', \hat{k}')}{(2\pi)^3 (\alpha^2 + q'^2) \left( \frac{q_0^2}{4} - q_0'^2 \right)}. \end{aligned} \quad (\text{VI-21})$$

The angular integral implied in  $d\bar{q}'$  is obtained from orthonormality of the spherical harmonics, from which Eq. (VI-21) becomes

$$\int \frac{d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}' \cdot \bar{r}} q' Y_{l'}^{\beta}(\hat{q}', \hat{k}')}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{q_0^2}{4} - q'^2\right)} = 4\pi (2\pi)^{-3} i Y_{l'}^{\beta}(\hat{r}, \hat{k}') (\alpha^2 + q'^2) \int_0^{\infty} \frac{dq' q'^3 j_{l'}(q'r)}{(\alpha^2 + q'^2) \left(\frac{q_0^2}{4} - q'^2\right)}. \quad (\text{VI-22})$$

This result can now be identified with the intermediate integrals defined in Eq. (2-24). We have

$$\int \frac{d\bar{q}' (\alpha^2 + q'^2) e^{i\bar{q}' \cdot \bar{r}} q' Y_{l'}^{\beta}(\hat{q}', \hat{k}')}{(2\pi)^3 (\alpha^2 + q'^2) \left(\frac{q_0^2}{4} - q'^2\right)} = (4\pi)^{-1} \mu^2 i Y_{l'}^{\beta}(\hat{r}, \hat{k}') \tilde{f}_{3l'}^{\sim}(r). \quad (\text{VI-23})$$

The final-state nuclear momentum operator is given by

$$\bar{k}' \rightarrow +i \bar{\nabla}_r. \quad (\text{VI-24})$$

Tensor components of this gradient operator are identified by Eq. (VI-6). We make use of Eqs. (VI-23) and (VI-24) to write the transition operator in Eq. (VI-20) as



$$\begin{aligned}
T^4 &= -\lambda_+ (4\pi)^{3/2} f \mu^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} e^{i\vec{q}\cdot\vec{R}} \left( \tau_{12}^+ \tau_{12}^3 - \tau_{12}^3 \tau_{12}^+ \right) \\
&\sum_{\alpha\beta\gamma} (-1)^\alpha \epsilon_{\alpha\beta\gamma} \left( e^{-i\vec{q}\cdot\vec{r}} \sigma_1^\alpha \sigma_2^{-\alpha} - e^{+i\vec{q}\cdot\vec{r}} \sigma_1^\alpha \sigma_2^{-\alpha} \right) \frac{q^\gamma}{\mu} \\
&\frac{q_0}{M} \sqrt{\frac{4\pi}{3}} (4\pi)^{-1} Y_{11}^\beta(r, \hat{k}') \tilde{f}_{31}(r) \nabla_{r'}^\beta. \quad (\text{VI-25})
\end{aligned}$$

The external pion wave functions can be expanded into partial waves through the relationships,

$$e^{\pm i\vec{q}\cdot\vec{r}} = 4\pi \sum_{\lambda\mu} i^\lambda (\pm 1)^\mu Y_{\lambda\mu}^*(\hat{q}, \hat{k}') Y_{\lambda\mu}^\eta(r, \hat{k}') j_\lambda\left(\frac{qr}{2}\right). \quad (\text{VI-26})$$

Then the factor in Eq. (VI-25) which contains the spin operators can be written

$$\begin{aligned}
&\left( e^{-i\vec{q}\cdot\vec{r}} \sigma_1^\alpha \sigma_2^{-\alpha} - e^{+i\vec{q}\cdot\vec{r}} \sigma_1^\alpha \sigma_2^{-\alpha} \right) = \\
&4\pi \sum_{\lambda\mu} i^\lambda (-1)^\mu Y_{\lambda\mu}^*(\hat{q}, \hat{k}') Y_{\lambda\mu}^\eta(r, \hat{k}') j_\lambda\left(\frac{qr}{2}\right) \left( \sigma_1^\alpha \sigma_2^{-\alpha} - (-1)^\mu \sigma_1^\alpha \sigma_2^{-\alpha} \right). \quad (\text{VI-27})
\end{aligned}$$

We need also to express the term,  $q^\gamma/\mu$ , in Eq. (VI-25) as a spherical harmonic; according to Eq. (VI-5), the required relationship is

$$\frac{q^{\delta}}{\mu} = \sqrt{\frac{4\pi}{3}} \frac{q}{\mu} Y_1^{\delta}(\hat{q}, \hat{k}'). \quad (\text{VI-28})$$

This spherical harmonic must be contracted with that of similar argument in Eq. (VI-27). This product can be written as

$$Y_{-1}^{*2}(\hat{q}, \hat{k}') \frac{q^{\delta}}{\mu} = \sqrt{\frac{4\pi}{3}} \frac{q}{\mu} (-1)^2 Y_{-1}^{-2}(\hat{q}, \hat{k}') Y_1^{\delta}(\hat{q}, \hat{k}'). \quad (\text{VI-29})$$

In obtaining this result, we have also made use of the relationship,

$$Y_{l_1}^{*m_1}(\Omega) = (-1)^{m_1} Y_{l_1}^{-m_1}(\Omega). \quad (\text{VI-30})$$

The product of two spherical harmonics may be contracted through use of the expression,

$$Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega) = \sum_{m_3, l_3} Y_{l_3}^{m_3}(\Omega) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} \begin{bmatrix} l_3 & 0 \\ l_1 & 0 \\ l_2 & 0 \end{bmatrix} \begin{bmatrix} l_3 & m_3 \\ l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}. \quad (\text{VI-31})$$

Equation (VI-29) can therefore be written as

$$\begin{aligned}
Y_{\lambda}^{\eta}(\hat{q}, \hat{k}') \frac{q^{\gamma}}{\mu} &= \\
\sqrt{\frac{4\pi}{3}} \frac{q}{\mu} (-1)^{\eta} \sum_{\lambda\mu} Y_{\lambda}^{-\mu}(\hat{q}, \hat{k}') \sqrt{\frac{(2\Lambda+1)3}{4\pi(2\lambda+1)}} \begin{bmatrix} \lambda & 0 \\ \Lambda & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda-\mu \\ \Lambda-\eta \\ 1 \ \gamma \end{bmatrix}. & \quad (\text{VI-32})
\end{aligned}$$

Using this result, the product of  $q^{\gamma}/\mu$  and Eq. (VI-27) can be written as

$$\begin{aligned}
\left( e^{-i\frac{\gamma}{2}\bar{r}} \sigma_1^{\gamma} \sigma_2^{\alpha} - e^{+i\frac{\gamma}{2}\bar{r}} \sigma_1^{\alpha} \sigma_2^{-\gamma} \right) \frac{q^{\gamma}}{\mu} &= \\
4\pi \sum_{\eta\Lambda} i^{\Lambda} (-1)^{\Lambda} Y_{\Lambda}^{\eta}(\hat{r}, \hat{k}') j_{\Lambda}\left(\frac{qr}{2}\right) \left( \sigma_1^{-\gamma} \sigma_2^{\alpha} - (-1)^{\Lambda} \sigma_1^{\alpha} \sigma_2^{-\gamma} \right) & \\
\frac{q}{\mu} (-1)^{\eta} \sum_{\lambda\mu} (-1)^{\mu} Y_{\lambda}^{*\mu}(\hat{q}, \hat{k}') \sqrt{\frac{2\Lambda+1}{2\lambda+1}} \begin{bmatrix} \lambda & 0 \\ \Lambda & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda-\mu \\ \Lambda-\eta \\ 1 \ \gamma \end{bmatrix}. & \quad (\text{VI-33})
\end{aligned}$$

Use has again been made of Eq. (VI-30) in obtaining this result.

We desire to retain the quantum numbers,  $\mu$  and  $\lambda$ , on the spherical harmonic of argument,  $(\hat{q}, \hat{k}')$ , and therefore interchange the order of evaluation of the sums on  $\eta$  and  $\Lambda$  with those on  $\mu$  and  $\lambda$ . From application of the sum rule to the last Clebsch-Gordan coefficient, it is seen that only the projection,  $\eta = \mu + \gamma$ , can contribute. By the triangle rule, the sum over  $\Lambda$  need include only the values,  $\lambda-1$ ,  $\lambda$  and  $\lambda+1$ . The three required zero-projection coupling coefficients are

$$\begin{bmatrix} \lambda & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} = \sqrt{\frac{\lambda}{2\lambda-1}},$$

$$\begin{bmatrix} \lambda & 0 \\ \lambda & 0 \\ 1 & 0 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} \lambda & 0 \\ \lambda+1 & 0 \\ 1 & 0 \end{bmatrix} = -\sqrt{\frac{\lambda+1}{2\lambda+3}}.$$

(VI-34)

Making use of these results, Eq. (VI-33) can then be written as

$$\begin{aligned} & \left( e^{-i\frac{\bar{q}\cdot\bar{r}}{2}} \sigma_1 \sigma_2^{-\alpha} - e^{+i\frac{\bar{q}\cdot\bar{r}}{2}} \sigma_1 \sigma_2^{\alpha} \right) \frac{q^\delta}{\mu} = \\ & 4\pi \sum_{\lambda\mu} (i)^{\lambda+1} (-1)^\lambda (-1)^\delta \frac{q}{\mu} Y_{\lambda}^{\mu*}(\hat{q}, \hat{k}') (\sigma_1 \sigma_2^{-\alpha} + (-1)^\lambda \sigma_1 \sigma_2^{\alpha}) \\ & \left\{ \sqrt{\frac{\lambda}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda-1}^{\mu+\delta}(\hat{r}, \hat{k}') j_{\lambda-1}^i\left(\frac{qr}{2}\right) \right. \\ & \left. + \sqrt{\frac{\lambda+1}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda+1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda+1}^{\mu+\delta}(\hat{r}, \hat{k}') j_{\lambda+1}^i\left(\frac{qr}{2}\right) \right\}. \end{aligned}$$

(VI-35)

The transition operator in Eq. (VI-25) is therefore given by

$$\begin{aligned}
T^{\lambda'} &= - \lambda_+ (4\pi)^{3/2} f \bar{\mu}' \sqrt{\frac{1}{2q_0}} \frac{1}{\sqrt{V}} e^{i\vec{q}\cdot\vec{R}} \begin{pmatrix} +3 & + \\ \tau_1 & \tau_2 \end{pmatrix} \\
&\sum_{\substack{\mu\lambda q \\ \alpha\beta\delta}} i^{\lambda+1} \epsilon_{\alpha\beta\delta} (-1)^{q+\delta} Y_{\lambda}^{*\mu}(\hat{k}, \hat{q}) \left( \sigma_1^{\alpha} \sigma_2^{\alpha} + (-1)^{\lambda} \sigma_1^{\alpha} \sigma_2^{\alpha-q} \right) \\
&\left\{ \sqrt{\frac{\lambda}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda-1}^{\mu+\delta}(\hat{r}, \hat{k}') j_{\lambda-1} \left( \frac{qr}{2} \right) \right. \\
&\left. + \sqrt{\frac{\lambda+1}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda+1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda+1}^{\mu+\delta}(\hat{r}, \hat{k}') j_{\lambda+1} \left( \frac{qr}{2} \right) \right\} \\
&\frac{q}{\mu} \frac{q_0}{M} \sqrt{\frac{4\pi}{3}} Y_1^{\beta}(\hat{r}, \hat{k}') \tilde{f}_{31}^{\sim}(r) \nabla_{r'}^q . \tag{VI-36}
\end{aligned}$$

In Eq. (VI-36), the argument of the spherical harmonic,  $Y_{\lambda}^{\mu}(\hat{q}, \hat{k}')$ , has been reversed by use of the expression,

$$Y_{\ell_1}^{m_1}(\Omega) = (-1)^{\ell_1} Y_{\ell_1}^{m_1}(-\Omega) . \tag{VI-37}$$

This spherical harmonic provides the matrix element with a functional dependence on the angle between either outgoing proton and the incident pion.

We are now ready to evaluate the matrix element, which will be called  $M^{\lambda'}$ . The initial state wave function is given by Eq. (2-8), while the final state function is given by Eq. (2-9). Both are of the form,

$$\psi = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{R}} \frac{u}{r} |JMLSTM_T\rangle. \quad (\text{VI-38})$$

Both states are thus assumed to be of good quantum numbers,  $J, M, L, S, T$  and  $M_T$ . Final state variables will be designated with prime superscripts. The state functions will be written symbolically as

$$|\psi\rangle = |J'M'L'S'T'M_T' u'K\rangle. \quad (\text{VI-39})$$

From Eq. (2-12), the component of the matrix element being calculated is

$$M^{4'} = \langle J'M'L'S'T'M_T' u'K | T^{4'} | JMLSTM_T uK \rangle. \quad (\text{VI-40})$$

As stated previously in Chapter 2, the integral over the center-of-mass coordinate,  $\vec{R}$ , simply yields a delta function in momentum, according to

$$(2\pi)^{-3} \int d\vec{R} e^{-i\vec{k}'\cdot\vec{R}} e^{i\vec{k}\cdot\vec{R}} e^{i\vec{q}\cdot\vec{R}} = \delta(\vec{k} + \vec{q} - \vec{k}'). \quad (\text{VI-41})$$

Total momentum is conserved in the calculation. The matrix element is evaluated in the overall-center-of-mass frame, for which  $\vec{K}' = 0$ .

Evaluation of the isospin matrix element is also straightforward, using the results in Eq. (VI-3). The isospin contribution from  $T^{4'}$  is

$$\langle 1, 1 | \hat{T}_1^+ \hat{T}_2^3 - \hat{T}_1^3 \hat{T}_2^+ | 0, 0 \rangle = -2. \quad (\text{VI-42})$$

In performing the integration over the relative coordinate,  $\vec{r}$ , we obtain the matrix element associated with a single spherical harmonic,  $Y_{\lambda}^{\mu}(\hat{k}, \hat{q})$ . This provides a term with definite functional dependence on  $\theta$ , the angle between outgoing proton and incident pion. The full matrix element (for the indicated nuclear states) is then determined by summing over the quantum number,  $\lambda$ . The projection,  $\mu$ , is found to be constrained by the projections,  $M$  and  $M'$ . The required sums over these projections will be performed in the evaluation of the cross section. The matrix element under consideration can now be written as

$$\begin{aligned} m^{4'} = & \sum_{\lambda \mu} (4\pi)^{3/2} f_{\mu}^{-1} \frac{1}{\sqrt{2g_0}} \frac{1}{\sqrt{V}} i^{\lambda+1} Y_{\lambda}^{*\mu}(\hat{k}', \hat{q}) \\ & \lambda_4 2 \frac{g}{\mu} \frac{g_0}{M} \langle J' M' L' S' g' | \int_{31}^{\sim}(r) \sum_{g \alpha \beta \delta} \epsilon_{\alpha \beta \delta} (-1)^{g+\delta} \\ & (\sigma_1^{-g} \sigma_2^{\alpha} + (-1)^{\lambda} \sigma_1^{\alpha} \sigma_2^{-g}) \sqrt{\frac{4\pi}{3}} \left\{ \sqrt{\frac{\lambda}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda-1}^{\mu+\delta}(r, \hat{k}') j_{\lambda-1}(\frac{gr}{2}) \right. \\ & \left. + \sqrt{\frac{\lambda+1}{2\lambda+1}} \begin{bmatrix} \lambda & -\mu \\ \lambda+1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} Y_{\lambda+1}^{\mu+\delta}(r, \hat{k}') j_{\lambda+1}(\frac{gr}{2}) \right\} Y_{1}^{\beta}(r, \hat{k}') \nabla_{\vec{r}}^g | J M L S U \rangle. \end{aligned} \quad (\text{VI-43})$$

All momentum variables within the bracket in Eq. (VI-43) are measured with respect to the vector,  $\hat{k}'$ . For brevity, these arguments will be dropped in subsequent equations.

Next, the initial- and final-state vectors are expanded into components, according to

$$|JMLSu\rangle = \sum_{M_L M_S} \begin{bmatrix} J & M \\ L & M_L \\ S & M_S \end{bmatrix} |LM_Lu\rangle |SM_S\rangle. \quad (\text{VI-44})$$

It is then possible to make an identification with the spin matrix element defined in Eq. (VI-13). The spin contribution in Eq. (VI-43) is

$$\langle S'M'_S | \sigma_1^{g\alpha} \sigma_2^{-g\alpha} + (-1)^{\lambda} \sigma_1^{\lambda\alpha} \sigma_2^{-g\alpha} | SM_S \rangle = F_{-g\alpha}. \quad (\text{VI-45})$$

Calculation of this term is performed using Eq. (VI-14). With the above results, Eq. (VI-43) can be written as

$$\begin{aligned} m^{g'} &= \sum_{\lambda\mu} (4\pi)^{3/2} f_{\mu}^{-1} \frac{1}{\sqrt{2g_0}} \frac{1}{\sqrt{V}} i^{\lambda+1} Y_{\lambda}^{*\mu}(\hat{k}^{\wedge}) \\ &\left\{ \frac{\sqrt{\lambda}}{\sqrt{2\lambda+1}} \sqrt{\frac{4\pi}{3}} \sum_{\substack{\alpha\beta\delta g \\ M'_L M'_S M_L M_S}} (-1)^{g+\delta} E_{\alpha\beta\delta} F_{-g\alpha} \begin{bmatrix} J' M' \\ L' M'_L \\ S' M'_S \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} \right. \\ &\lambda_+ 2 \frac{g}{\mu} \frac{g_0}{M} \langle L' M'_L g' | f_{31} \tilde{f}_{\lambda-1} Y_{\lambda-1}^{\mu+\delta} Y_1^{\beta} \nabla_{r'}^g | L M_L u \rangle \\ &\left. + \frac{\sqrt{\lambda+1}}{\sqrt{2\lambda+1}} \sqrt{\frac{4\pi}{3}} \sum_{\substack{\alpha\beta\delta g \\ M'_L M'_S M_L M_S}} (-1)^{g+\delta} E_{\alpha\beta\delta} F_{-g\alpha} \begin{bmatrix} J' M' \\ L' M'_L \\ S' M'_S \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda+1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} \right\} \end{aligned}$$



$$\lambda_+ 2 \frac{q}{\mu} \frac{q_0}{M} \left\langle L' M'_L g' \left| \tilde{f}_{31} \tilde{f}_{\lambda+1} Y_{\lambda+1}^{\mu+\delta} Y_1^\beta \nabla_{r'}^q \right| L M_L u \right\rangle \left. \right\}.$$

(VI-46)

The gradient,  $\tilde{\nabla}_{r'}$ , is meant to operate on the final nuclear state. Therefore it is necessary to contract the initial-state orbital angular momentum with two other spherical harmonics in the operator. This is accomplished by repetitive use of Eq. (VI-31). The result for the first bracket in Eq. (VI-46) is

$$Y_L^{M_L} Y_{\lambda-1}^{\mu+\delta} Y_1^\beta = \sum_{\ell \ell'} Y_\ell^{M_L + \mu + \delta + \beta} \frac{(2L+1)(2\lambda-1)3}{\sqrt{(4\pi)^2(2\ell+1)}} \begin{bmatrix} \ell & 0 \\ L & 0 \\ \ell' & 0 \end{bmatrix} \begin{bmatrix} \ell & M_L + \mu + \delta + \beta \\ L & M_L \\ \ell' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} \ell' & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \ell' & \mu + \delta + \beta \\ \lambda-1 & \mu + \delta \\ 1 & \beta \end{bmatrix}$$

(VI-47)

Making use of this relation and Eq. (VI-30), the first bracket in Eq. (VI-46) can be written as

$$\left\langle L' M'_L g' \left| \tilde{f}_{31} \tilde{f}_{\lambda-1} Y_{\lambda-1}^{\mu+\delta} Y_1^\beta \nabla_{r'}^q \right| L M_L u \right\rangle = \sum_{\ell \ell'} \frac{(2L+1)(2\lambda-1)3}{\sqrt{(4\pi)^2(2\ell+1)}} \begin{bmatrix} \ell & 0 \\ L & 0 \\ \ell' & 0 \end{bmatrix} \begin{bmatrix} \ell & M_L + \mu + \delta + \beta \\ L & M_L \\ \ell' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} \ell' & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \ell' & \mu + \delta + \beta \\ \lambda-1 & \mu + \delta \\ 1 & \beta \end{bmatrix} (-1)^{M'_L + M_L + \mu + \delta + \beta} \left\langle \ell, -(M_L + \mu + \delta + \beta), u \left| \tilde{f}_{31} \tilde{f}_{\lambda-1} \nabla_{r'}^q \right| L' M'_L g'^* \right\rangle.$$

(VI-48)

It is now possible to make use of the Wigner-Eckart theorem<sup>80</sup> to evaluate this bracket. The gradient is a spherical tensor of rank one, so that the general result needed is

$$\langle l_2 m_2 | \nabla^g | l_1 m_1 \rangle = \begin{bmatrix} l_2 & m_2 \\ l_1 & m_1 \\ 1 & g \end{bmatrix} \langle l_2 || \nabla_1 || l_1 \rangle. \quad (\text{VI-49})$$

The reduced matrix element is determined by evaluating the left-hand side of this equation for the case,  $g = m_1 = m_2 = 0$ . The reduced matrix element is then given by

$$\langle l_2 || \nabla_1 || l_1 \rangle = \frac{\langle l_2 0 | \nabla^0 | l_1 0 \rangle}{\begin{bmatrix} l_2 & 0 \\ l_1 & 0 \\ 1 & 0 \end{bmatrix}}. \quad (\text{VI-50})$$

The required matrix element in Eq. (VI-50) can be written in integral form as

$$\langle l_2 0 | \nabla^0 | l_1 0 \rangle = \int d\Omega Y_{l_2}^{*0} \nabla^0 Y_{l_1}^0. \quad (\text{VI-51})$$

In spherical polar coordinates, the gradient operator is given by

$$\bar{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \quad (\text{VI-52})$$

so that the component in the z-direction is just

$$\nabla^{\circ} = \cos\theta \frac{\partial}{\partial r} - \sin\theta \frac{1}{r} \frac{\partial}{\partial \theta} . \quad (\text{VI-53})$$

Making use of this relation and the identities,<sup>116</sup>

$$(\ell-m+1)P_{\ell+1}^m - (2\ell+1)\cos\theta P_{\ell}^m + (\ell+m)P_{\ell-1}^m = 0$$

and

$$-\sin\theta \frac{d}{d\theta} P_{\ell}^m = (\ell+m)P_{\ell-1}^m - \ell \cos\theta P_{\ell}^m , \quad (\text{VI-54})$$

the integral in Eq. (VI-51) is finally obtained by orthonormality. The result is

$$\int d\Omega Y_{\ell_2}^{\circ} \nabla^{\circ} Y_{\ell_1}^{\circ} = \delta_{\ell_2, \ell_1+1} \frac{\ell_1+1}{\sqrt{(2\ell_1+1)(2\ell_1+3)}} \left( \frac{d}{dr} - \frac{\ell_1}{r} \right) + \delta_{\ell_2, \ell_1-1} \frac{\ell_1}{\sqrt{(2\ell_1+1)(2\ell_1-1)}} \left( \frac{d}{dr} + \frac{\ell_1+1}{r} \right) . \quad (\text{VI-55})$$

The required Clebsch-Gordan coefficients in Eq. (VI-50) are

$$\begin{bmatrix} \ell_1+1 & 0 \\ \ell_1 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\sqrt{\ell_1+1}}{\sqrt{2\ell_1+1}}$$

and

$$\begin{bmatrix} l_1-1 & 0 \\ l_1 & 0 \\ 1 & 0 \end{bmatrix} = -\sqrt{\frac{l_1}{2l_1+1}} \quad (\text{VI-56})$$

With the results of Eqs. (VI-55) and (VI-56), the reduced matrix element in Eq. (VI-50) becomes

$$\langle l_2 \| \nabla_r \| l_1 \rangle = \delta_{l_2, l_1+1} \sqrt{\frac{l_1+1}{2l_1+3}} \left( \frac{d}{dr} - \frac{l_1}{r} \right) - \delta_{l_2, l_1-1} \sqrt{\frac{l_1}{2l_1-1}} \left( \frac{d}{dr} + \frac{l_1+1}{r} \right) \quad (\text{VI-57})$$

Use is made of this general result to write Eq. (VI-48) as

$$\begin{aligned} & \langle L' M'_L g' | \tilde{f}_{31}^j Y_{\lambda-1}^{\mu+\delta} Y_1^\beta \nabla_r^q | L M_L u \rangle = \\ & (-1)^{M'_L + M_L + \mu + \delta + \beta} \sqrt{\frac{(2L+1)(2\lambda-1)3}{(4\pi)^2}} \sum_{L'=|L-2|}^L \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\delta+\beta \\ \lambda-1 & \mu+\delta \\ 1 & \beta \end{bmatrix} \\ & \left\{ \begin{bmatrix} L'+1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'+1 & M'_L + \mu + \delta + \beta \\ L & M_L \\ L' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} L'+1 - (M'_L + \mu + \delta + \beta) \\ L' & -M'_L \\ 1 & q \end{bmatrix} \frac{\sqrt{L'+1}}{2L'+3} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^j \left( \frac{d}{dr} - \frac{L'}{r} \right) g'^* \right. \\ & \left. - \begin{bmatrix} L'-1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'-1 & M'_L + \mu + \delta + \beta \\ L & M_L \\ L' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} L'-1 - (M'_L + \mu + \delta + \beta) \\ L' & -M'_L \\ 1 & q \end{bmatrix} \frac{\sqrt{L'}}{2L'-1} \int dr r^2 \frac{u}{r} \tilde{f}_{31}^j \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g'^* \right\} \quad (\text{VI-58}) \end{aligned}$$

A similar result is obtained for the second bracket (involving  $Y_{\lambda+1}^{\mu+\gamma}$ ) in Eq. (VI-46).

Finally, the matrix element is obtained by use of Eq. (VI-58) in Eq. (VI-46). Upon rearrangement, the result is

$$\begin{aligned}
 \mathcal{M}^{4'} &= \sum_{\lambda\mu} (4\pi)^{3/2} f_{\mu}^{-1} \sqrt{\frac{1}{2g_0}} \frac{1}{\sqrt{V}} i^{\lambda+1} Y_{\lambda}^{\mu} \left( \hat{R}, \hat{g} \right) \\
 &\left[ \frac{\sqrt{\lambda}}{\sqrt{2\lambda+1}} \sqrt{\frac{4\pi}{3}} \sum_{\substack{\alpha\beta\delta\gamma \\ M_L' M_L M_L M_S}} (-1)^{\gamma+\delta} E_{\alpha\beta\delta}^{-\gamma\alpha} F \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix} \right] \\
 &(-1)^{M_L'+M_L+\mu+\delta+\beta} \sqrt{\frac{(2L+1)(2\lambda-1)3}{(4\pi)^2}} \sum_{L'=1\lambda-2l}^{\lambda} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\delta+\beta \\ \lambda-1 & \mu+\delta \\ 1 & \beta \end{bmatrix} \\
 &\left\{ \begin{bmatrix} L'+1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'+1 & M_L'+\mu+\delta+\beta \\ L & M_L \\ L' & \mu+\delta+\beta \end{bmatrix} \begin{bmatrix} L'+1 & -(M_L'+\mu+\delta+\beta) \\ L' & -M_L' \\ 1 & \gamma \end{bmatrix} \frac{\sqrt{L'+1}}{2L'+3} \right. \\
 &\quad \left. \lambda_+ 2 \frac{\gamma}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{3l} \tilde{f}_{\lambda-1} \left( \frac{d}{dr} - \frac{L'}{r} \right) g_i^* \right. \\
 &\quad \left. - \begin{bmatrix} L'-1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'-1 & M_L'+\mu+\delta+\beta \\ L & M_L \\ L' & \mu+\delta+\beta \end{bmatrix} \begin{bmatrix} L'-1 & -(M_L'+\mu+\delta+\beta) \\ L' & -M_L' \\ 1 & \gamma \end{bmatrix} \frac{\sqrt{L'}}{2L'-1} \right. \\
 &\quad \left. \lambda_+ 2 \frac{\gamma}{\mu} \frac{g_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{3l} \tilde{f}_{\lambda-1} \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g_i^* \right\} \\
 &+ \sqrt{\frac{\lambda+1}{2\lambda+1}} \sqrt{\frac{4\pi}{3}} \sum_{\substack{\alpha\beta\delta\gamma \\ M_L' M_L M_L M_S}} (-1)^{\gamma+\delta} E_{\alpha\beta\delta}^{-\gamma\alpha} F \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda+1 & -(\mu+\delta) \\ 1 & \delta \end{bmatrix}
 \end{aligned}$$

$$(-1)^{M'_L + M_L + \mu + \delta + \beta} \sqrt{\frac{(2L+1)(2\lambda+3)3}{(4\pi)^2}} \sum_{L'=\lambda}^{\lambda+2} \begin{bmatrix} L' & 0 \\ \lambda+1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu + \delta + \beta \\ \lambda+1 & \mu + \delta \\ 1 & \beta \end{bmatrix}$$

$$\left\{ \begin{bmatrix} L'+1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'+1 & M_L + \mu + \delta + \beta \\ L & M_L \\ L' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu + \delta + \beta) \\ L' & -M'_L \\ 1 & \beta \end{bmatrix} \frac{\sqrt{L'+1}}{2L'+3} \right.$$

$$\lambda_+ 2 \frac{\beta}{\mu} \frac{\beta_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{\beta'}^i j_{\lambda+1}^i \left( \frac{d}{dr} - \frac{L'}{r} \right) g^i$$

$$- \begin{bmatrix} L'-1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'-1 & M_L + \mu + \delta + \beta \\ L & M_L \\ L' & \mu + \delta + \beta \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu + \delta + \beta) \\ L' & -M'_L \\ 1 & \beta \end{bmatrix} \frac{\sqrt{L'}}{2L'-1}$$

$$\left. \lambda_+ 2 \frac{\beta}{\mu} \frac{\beta_0}{M} \int dr r^2 \frac{u}{r} \tilde{f}_{\beta'}^i j_{\lambda+1}^i \left( \frac{d}{dr} + \frac{L'+1}{r} \right) g^i \right\}$$

(VI-59)

With the definition provided in Eq. (2-22), the top line in Eq. (VI-59) is common to all matrix elements. The remaining factors (in square brackets) are divided into Clebsch-Gordan products,  $R_i$ , and radial integrals,  $I_i$ . In particular, the matrix element defined in Eq. (VI-59) is represented in this scheme as integrals 21 through 24. That is, this component of the matrix element can be written as

$$m^{\lambda'} = \sum_{\lambda\mu} (4\pi)^{3/2} f_{\mu}^{-1} \frac{1}{\sqrt{2\beta_0}} \frac{1}{\sqrt{V}} \cdot \lambda+1 Y_{\lambda}^{*\mu}(\hat{R}, \hat{\beta}) \sum_{J'L'L} \sum_{i=21}^{24} R_i I_i, \quad (\text{VI-60})$$

where  $R_i$  and  $I_i$  are, of course, also functions of  $M$  and  $M'$ . This is the

form that motivates Eqs. (2-13) and (2-22). The projection of the spherical harmonic in Eq. (2-13) is the ultimate result of angular momentum conservation.

The radial integrals,  $I_i$ , depend on the Lagrangian formalism; therefore different integrals are defined in Chapters 2, 3 and 4. The angular coefficients,  $R_i$ , are common to all three formalisms, and these results are summarized below. All coefficients have been found to satisfy a Wigner-Eckart result involving conservation of angular momentum for the absorption process. The coefficients have been evaluated numerically over a large set of quantum numbers without a violation of this result. Projection and principle sums in the coefficients are stated explicitly (or omitted); in practice, these were reduced wherever practical with the sum and triangle rules. The coefficients are:

$$R_1 = +(-1)^\lambda \sqrt{\frac{(2L'+1)(2L+1)(\lambda+1)}{4\pi}} \frac{1}{2\lambda+3} \begin{bmatrix} \lambda+1 & 0 \\ L' & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{M_L+q} S_{-q} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda+1 & M_L-M_L' \\ L' & -M_L' \\ L & M_L \end{bmatrix} \begin{bmatrix} \lambda+1 & -(M_L-M_L') \\ \lambda & \mu \\ 1 & q \end{bmatrix},$$

$$R_2 = -(-1)^\lambda \sqrt{\frac{(2L'+1)(2L+1)\lambda}{4\pi}} \frac{1}{2\lambda-1} \begin{bmatrix} \lambda-1 & 0 \\ L' & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{M_L+q} S_{-q} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda-1 & M_L-M_L' \\ L' & -M_L' \\ L & M_L \end{bmatrix} \begin{bmatrix} \lambda-1 & -(M_L-M_L') \\ \lambda & \mu \\ 1 & q \end{bmatrix},$$

$$R_3 = +(-1)^\lambda \sqrt{\frac{(2L+1)(2\lambda+1)(L'+1)}{4\pi}} \frac{1}{2L'+3} \begin{bmatrix} L'+1 & 0 \\ \lambda & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{M_L'+M_L+\mu+q} S_{-q} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} L'+1 & M_L+\mu \\ \lambda & \mu \\ L & M_L \end{bmatrix} \begin{bmatrix} L'+1-(M_L+\mu) \\ L' & -M_L' \\ 1 & q \end{bmatrix},$$

$$R_4 = -(-1)^\lambda \sqrt{\frac{(2L+1)(2\lambda+1)L'}{4\pi}} \frac{1}{2L'-1} \begin{bmatrix} L'-1 & 0 \\ \lambda & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{M_L'+M_L+\mu+q} S_{-q} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} L'-1 & M_L+\mu \\ \lambda & \mu \\ L & M_L \end{bmatrix} \begin{bmatrix} L'-1-(M_L+\mu) \\ L' & -M_L' \\ 1 & q \end{bmatrix},$$

$$R_5 = R_7 = (-1)^\lambda R_3, \quad R_6 = R_8 = (-1)^\lambda R_4,$$

$$R_9 = + \sqrt{\frac{(2L+1)\lambda(2\lambda-1)}{4\pi(3)(2\lambda+1)(2L'+1)}} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{\alpha} S_{-\alpha} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & \alpha \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1-(\alpha+\mu) \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} L' & M_L' \\ \lambda-1 & \alpha+\mu \\ L & M_L \end{bmatrix},$$

$$R_{10} = + \sqrt{\frac{(2L+1)(\lambda+1)(2\lambda+3)}{4\pi(3)(2L'+1)(2\lambda+1)}} \begin{bmatrix} L' & 0 \\ \lambda+1 & 0 \\ L & 0 \end{bmatrix}$$

$$\sum (-1)^{\alpha} S_{-\alpha} \begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & \alpha \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda+1-(\alpha+\mu) \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} L' & M_L' \\ \lambda+1 & \alpha+\mu \\ L & M_L \end{bmatrix},$$

$$R_{11} = + \sqrt{\frac{2(2L+1)\lambda(2\lambda-1)}{4\pi(3)(2L'+1)(2\lambda+1)}} \sum (-1)^{\alpha} S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1 & \beta-\mu \\ 1 & -\beta \end{bmatrix} \begin{bmatrix} 2 & \alpha+\beta \\ 1 & \alpha \\ 1 & \beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\alpha \\ \lambda-1 & \mu-\beta \\ 2 & \alpha+\beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ L' & 0 \\ L & 0 \end{bmatrix} \begin{bmatrix} L' & M_L' \\ L' & \alpha+\mu \\ L & M_L \end{bmatrix},$$



$$R_{12} = + \sqrt{\frac{2(2L+1)(\lambda+1)(2\lambda+3)}{4\pi(3)(2L'+1)(2\lambda+1)}} \sum (-1)^\alpha S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda + \beta - \mu \\ 1 - \beta \end{bmatrix} \begin{bmatrix} 2\lambda + \beta \\ 1 \alpha \\ 1 \beta \end{bmatrix} \begin{bmatrix} l' 0 \\ \lambda + 1 0 \\ 2 0 \end{bmatrix} \begin{bmatrix} l' \mu + \alpha \\ \lambda + 1 \mu - \beta \\ 2 \alpha + \beta \end{bmatrix} \begin{bmatrix} L' 0 \\ l' 0 \\ L 0 \end{bmatrix} \begin{bmatrix} L' M_L' \\ l' \alpha + \mu \\ L M_L \end{bmatrix} ,$$

$$R_{13} = + \sqrt{\frac{(2L+1)\lambda(2\lambda-1)(L'+1)}{4\pi(2\lambda+1)}} \frac{1}{2L'+3} \sum S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda - 1 \beta - \mu \\ 1 - \beta \end{bmatrix} \begin{bmatrix} l' 0 \\ \lambda - 1 0 \\ 1 0 \end{bmatrix} \begin{bmatrix} l' \mu \\ \lambda - 1 \mu - \beta \\ 1 \beta \end{bmatrix} \begin{bmatrix} L'+1 0 \\ L 0 \\ l' 0 \end{bmatrix} \begin{bmatrix} L'+1 M_L + \mu \\ L M_L \\ l' \mu \end{bmatrix} \begin{bmatrix} L'+1 - (M_L + \mu) \\ l' - M_L' \\ 1 \alpha \end{bmatrix} ,$$

$$R_{14} = - \sqrt{\frac{(2L+1)L'\lambda(2\lambda-1)}{4\pi(2\lambda+1)}} \frac{1}{2L'-1} \sum S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda - 1 \beta - \mu \\ 1 - \beta \end{bmatrix} \begin{bmatrix} l' 0 \\ \lambda - 1 0 \\ 1 0 \end{bmatrix} \begin{bmatrix} l' \mu \\ \lambda - 1 \mu - \beta \\ 1 \beta \end{bmatrix} \begin{bmatrix} L'-1 0 \\ L 0 \\ l' 0 \end{bmatrix} \begin{bmatrix} L'-1 M_L + \mu \\ L M_L \\ l' \mu \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu) \\ l' - M_L' \\ 1 \alpha \end{bmatrix} ,$$

$$R_{15} = + \sqrt{\frac{(2L+1)(L'+1)(\lambda+1)(2\lambda+3)}{4\pi(2\lambda+1)}} \frac{1}{2L'+3} \sum S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda + 1 \beta - \mu \\ 1 - \beta \end{bmatrix} \begin{bmatrix} l' 0 \\ \lambda + 1 0 \\ 1 0 \end{bmatrix} \begin{bmatrix} l' \mu \\ \lambda + 1 \mu - \beta \\ 1 \beta \end{bmatrix} \begin{bmatrix} L'+1 0 \\ L 0 \\ l' 0 \end{bmatrix} \begin{bmatrix} L'+1 M_L + \mu \\ L M_L \\ l' \mu \end{bmatrix} \begin{bmatrix} L'+1 - (M_L + \mu) \\ l' - M_L' \\ 1 \alpha \end{bmatrix} ,$$

$$R_{16} = - \sqrt{\frac{(2L+1)L'(\lambda+1)(2\lambda+3)}{4\pi(2\lambda+1)}} \frac{1}{2L'-1} \sum S_{-\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda + 1 \beta - \mu \\ 1 - \beta \end{bmatrix} \begin{bmatrix} l' 0 \\ \lambda + 1 0 \\ 1 0 \end{bmatrix} \begin{bmatrix} l' \mu \\ \lambda + 1 \mu - \beta \\ 1 \beta \end{bmatrix} \begin{bmatrix} L'-1 0 \\ L 0 \\ l' 0 \end{bmatrix} \begin{bmatrix} L'-1 M_L + \mu \\ L M_L \\ l' \mu \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu) \\ l' - M_L' \\ 1 \alpha \end{bmatrix} ,$$

$$R_{17} = + \sqrt{\frac{(2L+1)\lambda(2\lambda-1)}{4\pi(3)(2L'+1)(2\lambda+1)}} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ L & 0 \end{bmatrix} \sum (-1)^{\beta+\delta} \epsilon_{\alpha\beta\delta} F_{\beta\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -\beta \\ 1 & \beta \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1-(\mu+\delta) \\ 1 & \delta \end{bmatrix} \begin{bmatrix} L' & M_L' \\ \lambda-1 & \mu+\delta \\ L & M_L \end{bmatrix},$$

$$R_{18} = + \sqrt{\frac{(2L+1)(\lambda+1)(2\lambda+3)}{4\pi(3)(2L'+1)(2\lambda+1)}} \begin{bmatrix} L' & 0 \\ \lambda+1 & 0 \\ L & 0 \end{bmatrix} \sum (-1)^{\beta+\delta} \epsilon_{\alpha\beta\delta} F_{\beta\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -\beta \\ 1 & \beta \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda+1-(\mu+\delta) \\ 1 & \delta \end{bmatrix} \begin{bmatrix} L' & M_L' \\ \lambda+1 & \mu+\delta \\ L & M_L \end{bmatrix},$$

$$R_{19} = + \sqrt{\frac{2(2L+1)\lambda(2\lambda-1)}{4\pi(3)(2L'+1)(2\lambda+1)}} \sum (-1)^{g+\delta} \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1-(\mu+\delta) \\ 1 & \delta \end{bmatrix} \begin{bmatrix} 2 & g+\beta \\ 1 & g \\ 1 & \beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\delta+g+\beta \\ \lambda-1 & \mu+\delta \\ 2 & g+\beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ L' & M_L' \\ L & 0 \\ L & M_L \end{bmatrix},$$

$$R_{20} = + \sqrt{\frac{2(2L+1)(\lambda+1)(2\lambda+3)}{4\pi(3)(2L'+1)(2\lambda+1)}} \sum (-1)^{g+\delta} \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda+1-(\mu+\delta) \\ 1 & \delta \end{bmatrix} \begin{bmatrix} 2 & g+\beta \\ 1 & g \\ 1 & \beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ \lambda+1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\delta+g+\beta \\ \lambda+1 & \mu+\delta \\ 2 & g+\beta \end{bmatrix} \begin{bmatrix} L' & 0 \\ L' & M_L' \\ L & 0 \\ L & M_L \end{bmatrix},$$

$$R_{21} = + \sqrt{\frac{(2L+1)(L'+1)\lambda(2\lambda-1)}{4\pi(2\lambda+1)}} \frac{1}{2L'+3} \sum (-1)^{\delta} \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J' M' \\ L' M_L' \\ S' M_S' \end{bmatrix} \begin{bmatrix} J M \\ L M_L \\ S M_S \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \lambda-1-(\mu+\delta) \\ 1 & \delta \end{bmatrix} \begin{bmatrix} L'+1-(M_L'+\mu+\delta+\beta) \\ L' & -M_L' \\ 1 & g \end{bmatrix} \begin{bmatrix} L' & 0 \\ \lambda-1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L' & \mu+\delta+\beta \\ \lambda-1 & \mu+\delta \\ 1 & \beta \end{bmatrix} \begin{bmatrix} L'+1 & 0 \\ L & 0 \\ L' & 0 \end{bmatrix} \begin{bmatrix} L'+1 & M_L'+\mu+\delta+\beta \\ L & M_L \\ L' & \mu+\delta+\beta \end{bmatrix},$$

$$R_{22} = -\sqrt{\frac{(2L+1)L'\lambda(2\lambda-1)}{4\pi(2\lambda+1)}} \frac{1}{2L'-1} \sum (-1)^\delta \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J'M' \\ L'M'_L \\ S'M'_S \end{bmatrix} \begin{bmatrix} JM \\ LM_L \\ SM_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda - 1 - (\mu + \delta) \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu + \delta + \beta) \\ L' \quad -M'_L \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L' \quad 0 \\ \lambda - 1 \quad 0 \\ 1 \quad 0 \end{bmatrix} \begin{bmatrix} L' \mu + \delta + \beta \\ \lambda - 1 \quad \mu + \delta \\ 1 \quad \beta \end{bmatrix} \begin{bmatrix} L'-1 \quad 0 \\ L \quad 0 \\ L' \quad 0 \end{bmatrix} \begin{bmatrix} L'-1 M_L + \mu + \delta + \beta \\ L \quad M_L \\ L' \quad \mu + \delta + \beta \end{bmatrix},$$

$$R_{23} = +\sqrt{\frac{(2L+1)(L'+1)(\lambda+1)(2\lambda+3)}{4\pi(2\lambda+1)}} \frac{1}{2L'+3} \sum (-1)^\delta \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J'M' \\ L'M'_L \\ S'M'_S \end{bmatrix} \begin{bmatrix} JM \\ LM_L \\ SM_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda + 1 - (\mu + \delta) \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L'+1 - (M_L + \mu + \delta + \beta) \\ L' \quad -M'_L \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L' \quad 0 \\ \lambda + 1 \quad 0 \\ 1 \quad 0 \end{bmatrix} \begin{bmatrix} L' \mu + \delta + \beta \\ \lambda + 1 \quad \mu + \delta \\ 1 \quad \beta \end{bmatrix} \begin{bmatrix} L'+1 \quad 0 \\ L \quad 0 \\ L' \quad 0 \end{bmatrix} \begin{bmatrix} L'+1 M_L + \mu + \delta + \beta \\ L \quad M_L \\ L' \quad \mu + \delta + \beta \end{bmatrix}$$

and

$$R_{24} = -\sqrt{\frac{(2L+1)L'(\lambda+1)(2\lambda+3)}{4\pi(2\lambda+1)}} \frac{1}{2L'-1} \sum (-1)^\delta \epsilon_{\alpha\beta\delta} F_{-g\alpha}$$

$$\begin{bmatrix} J'M' \\ L'M'_L \\ S'M'_S \end{bmatrix} \begin{bmatrix} JM \\ LM_L \\ SM_S \end{bmatrix} \begin{bmatrix} \lambda - \mu \\ \lambda + 1 - (\mu + \delta) \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L'-1 - (M_L + \mu + \delta + \beta) \\ L' \quad -M'_L \\ 1 \quad \delta \end{bmatrix} \begin{bmatrix} L' \quad 0 \\ \lambda + 1 \quad 0 \\ 1 \quad 0 \end{bmatrix} \begin{bmatrix} L' \mu + \delta + \beta \\ \lambda + 1 \quad \mu + \delta \\ 1 \quad \beta \end{bmatrix} \begin{bmatrix} L'-1 \quad 0 \\ L \quad 0 \\ L' \quad 0 \end{bmatrix} \begin{bmatrix} L'-1 M_L + \mu + \delta + \beta \\ L \quad M_L \\ L' \quad \mu + \delta + \beta \end{bmatrix}.$$

(VI-61)

APPENDIX VII  
THE ANGULAR DISTRIBUTION

The matrix element expressed in Eq. (2-13) is cast as products of complex coefficients and spherical harmonics. The exact expression is

$$m = \sum_{\lambda=|M'-M|} C(S'M'M\lambda) Y_{\lambda}^{*M'-M}(\hat{k}, \hat{q}). \quad (\text{VII-1})$$

The differential cross section is proportional to the square of the matrix element, or

$$\frac{d\sigma_a}{d\Omega} \sim \sum_{S'M'M} |m|^2. \quad (\text{VII-2})$$

Making use of Eq. (VII-1), we get

$$\frac{d\sigma_a}{d\Omega} \sim \sum_{S'M'M} \sum_{\lambda, \lambda'=|M'-M|} C^*(S'M'M\lambda') C(S'M'M\lambda) Y_{\lambda'}^{*M'-M}(\hat{k}, \hat{q}) Y_{\lambda}^{*M'-M}(\hat{k}, \hat{q}). \quad (\text{VII-3})$$

The product of spherical harmonics can be contracted, yielding

$$Y_{\lambda'}^{*M'-M}(\hat{k}, \hat{q}) Y_{\lambda}^{*M'-M}(\hat{k}, \hat{q}) = (-1)^{M'-M} \sum_{\ell=|\lambda'-\lambda|}^{\lambda'+\lambda} Y_{\ell}^0(\hat{k}, \hat{q}) \sqrt{\frac{(2\lambda'+1)(2\lambda+1)}{4\pi(2\ell+1)}} \begin{bmatrix} \ell & 0 \\ \lambda' & 0 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \ell & 0 \\ \lambda' & M'-M \\ \lambda & M'-M \end{bmatrix}. \quad (\text{VII-4})$$

The single spherical harmonic can be written as a Legendre polynomial:

$$Y_{\ell}^0(\hat{k}, \hat{q}) = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos\theta). \quad (\text{VII-5})$$

It is in turn easily converted to a polynomial in  $\cos\theta$  through the relationship,<sup>117</sup>

$$P_{\ell}(\cos\theta) = a_{\ell}^{-1} \sum_{n=0}^{\ell} d_{\ell}^n \cos^n \theta. \quad (\text{VII-6})$$

With the use of Eqs. (VII-4), (VII-5) and (VII-6), the expression for the differential cross section in Eq. (VII-3) can be written as

$$\frac{d\sigma_a}{d\Omega} \sim \frac{1}{4\pi} \sum_{S'M'M} (-1)^{M'-M} \sum_{\lambda, \lambda'=|M'-M|} \sqrt{(2\lambda'+1)(2\lambda+1)} \\ C^*(S'M'M\lambda') C(S'M'M\lambda) \sum_{\ell=|\lambda'-\lambda|}^{\lambda'+\lambda} \begin{bmatrix} \ell & 0 \\ \lambda' & 0 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \ell & 0 \\ \lambda' & M'-M \\ \lambda & M-M' \end{bmatrix} a_{\ell}^{-1} \sum_{n=0}^{\ell} d_{\ell}^n \cos^n \theta. \quad (\text{VII-7})$$

This equation expresses the angular distribution as a polynomial in  $\cos\theta$ .

APPENDIX VIII  
PI-NUCLEON FREE SCATTERING AMPLITUDES

The wave function for pi-nucleon scattering is given by

$$\psi = \left\{ e^{i\vec{q}\cdot\vec{r}} + F \frac{e^{iqr}}{r} \right\} \chi, \quad (\text{VIII-1})$$

where  $\chi$  is the nucleon spin function and

$$F = \frac{1}{q} \sum_{l,t} (2l+1) f_{l,t} \Lambda_{l,t} \tau_t P_l(\cos\theta). \quad (\text{VIII-2})$$

In this expression, the quantities,  $f_{l,j,t}$ , are considered to be functions of orbital angular momentum,  $l$ , total angular momentum,  $j = l \pm \frac{1}{2}$ , and isospin,  $t$ . Identification may be made with the phase shifts through the relationship,

$$f_{l,t} = e^{i\delta_{l,t}} \sin\delta_{l,t}. \quad (\text{VIII-3})$$

In Eq. (VIII-2),  $\tau_t$  is a projection operator for isospin,  $t$ , while  $\Lambda_{l,j}$  is an angular momentum projection operator, given explicitly by

$$\Lambda_{l, l+1/2} = \frac{l+1 + \vec{l}\cdot\vec{\sigma}}{2l+1}$$

and

$$\Lambda_{l, l-1/2} = \frac{l - \vec{l}\cdot\vec{\sigma}}{2l+1}. \quad (\text{VIII-4})$$

The angular momentum operator is easily shown to be

$$\bar{l} = -i \hat{q}' \times \hat{q} \frac{d}{d(\cos \theta)} . \quad (\text{VIII-5})$$

An explicit sum over the first two partial waves and all possible values of  $j$  in Eq. (VIII-2) gives the result,

$$F = \sum_t \frac{1}{g} \left[ \left\{ f_{0\frac{1}{2}t} - f_{0\frac{1}{2}t} i \vec{\sigma} \cdot \hat{q}' \times \hat{q} \frac{d}{d(\cos \theta)} \right\} P_0(\cos \theta) + \left\{ 2f_{1\frac{3}{2}t} + f_{1\frac{1}{2}t} - (f_{1\frac{3}{2}t} - f_{1\frac{1}{2}t}) i \vec{\sigma} \cdot \hat{q}' \times \hat{q} \frac{d}{d(\cos \theta)} \right\} P_1(\cos \theta) \right] \tau_t . \quad (\text{VIII-6})$$

When the indicated operations on the Legendre polynomials are carried out, this equation reduces to

$$F = \sum_t \left[ S_{2t,1} + (2P_{2t,3} + P_{2t,1}) \hat{q}' \cdot \hat{q} + (P_{2t,1} - P_{2t,3}) i \vec{\sigma} \cdot \hat{q}' \times \hat{q} \right] \tau_t . \quad (\text{VIII-7})$$

In writing Eq. (VIII-7) use has been made of the symbols,

$$l_{2t,2t} = \frac{1}{g} f_{l,t,t} . \quad (\text{VIII-8})$$

Initial and final isospin product-states may be expanded in states of good total isospin by writing

$$|t_N \mu_N t_\pi \mu_\pi\rangle = \sum_{\substack{\mu = \mu_N + \mu_\pi \\ t = |t_N - t_\pi|}}^{t_N + t_\pi} \begin{bmatrix} t & \mu \\ t_N & \mu_N \\ t_\pi & \mu_\pi \end{bmatrix} |t \mu\rangle. \quad (\text{VIII-9})$$

Finally, the desired expression for the amplitude operator is obtained from the isospin matrix element,

$$\langle t'_N \mu'_N t'_\pi \mu'_\pi | F | t_N \mu_N t_\pi \mu_\pi \rangle. \quad (\text{VIII-10})$$

In particular, results are obtained for the following processes:

$$\pi^+ + p \rightarrow \pi^+ + p,$$

$$F = \left\{ S_{31} + (2P_{33} + P_{31}) \hat{q}'_B \cdot \hat{q}_B + (P_{31} - P_{33}) i\vec{\sigma} \cdot \hat{q}'_B \times \hat{q}_B \right\};$$

$$\pi^+ + n \rightarrow \pi^0 + p,$$

$$F = \frac{\sqrt{2}}{3} \left\{ (S_{31} - S_{11}) + (2P_{33} + P_{31} - 2P_{13} - P_{11}) \hat{q}'_B \cdot \hat{q}_B \right. \\ \left. + (P_{31} - P_{33} + P_{13} - P_{11}) i\vec{\sigma} \cdot \hat{q}'_B \times \hat{q}_B \right\};$$

$$\text{and } \pi^+ + n \rightarrow \pi^+ + n,$$

$$F = \frac{1}{3} \left\{ (2S_{11} + S_{31}) + (4P_{13} + 2P_{11} + 2P_{33} + P_{31}) \hat{q}'_B \cdot \hat{q}_B \right. \\ \left. + (2P_{11} - 2P_{13} + P_{31} - P_{33}) i\vec{\sigma} \cdot \hat{q}'_B \times \hat{q}_B \right\}. \quad (\text{VIII-11})$$



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